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利用 Laplace 变换求解分数阶 Allen-Cahn 方程

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摘要: 考虑 Caputo 型分数阶 Allen-Cahn 方程的高效数值算法, 利用 Laplace 变换将其转化为整数阶 Allen-Cahn 方程. 利用算子分裂方法进一步将其分解为热传导方程和非线性方程. 其中, 非线性方程精确求解, 热传导方程采用二阶差分方法求解. 数值实验表明了所给格式的有效性.

关键词: 分数阶 Allen-Cahn 方程; Caputo 型分数阶导数; Laplace 变换; 算子分裂; 能量递减

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Numerical Solution of Fractional Allen-Cahn Equation by Laplace Transform

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Abstract: An efficient numerical algorithm for Caputo-type fractional Allen-Cahn equation is considered. Firstly, the Laplace transform is used to transform it into integer order Allen-Cahn equation, and then the operator splitting method is used to decompose into heat conduction equation and nonlinear equation. The nonlinear equation is solved analytically and the heat conduction equation is solved using second-order finite difference method. Numerical experiments are presented to confirm the efficiency of the proposed method.

Keywords: fractional Allen-Cahn equation; Caputo-type fractional derivative; Laplace transform; operator splitting method; energy decline

1 预备知识

Allen-Cahn 方程是一类非齐次半线性泊松方程^[1], 是材料科学中描述流体动力学问题和反应扩散问题的一类重要方程. 在研究图像处理^[2]、平均曲率流量^[3]、晶体生长^[4]、人群扩散现象^[5]和随机扰动^[6]等问题时, Allen-Cahn 方程发挥着极为重要的作用.

考虑时间分数阶 Allen-Cahn 方程, 即

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$$\left. \begin{aligned} & {}^C_0D_t^\alpha u(x,t) + \frac{1}{\epsilon^2}u(u^2-1) - \frac{\partial^2 u}{\partial x^2} = 0, \quad (x,t) \in [a,b] \times (0,T], \\ & u(a,t) = f(t), \quad u(b,t) = g(t), \quad t \in (0,T], \\ & u(x,0) = u_0(x), \quad x \in [a,b]. \end{aligned} \right\} \tag{1}$$

式(1)中: ${}^C_0D_t^\alpha u(x,t)$ 为 Caputo 型分数阶导数. 即有

$${}^C_0D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 \leq \alpha < 1. \tag{2}$$

Allen-Cahn 方程可以视为 Lyapunov 能量泛函的 L_2 梯度流^[7]. 设基本能量泛函为 $E(u)$, 即

$$E(u) = \int_a^b \left[\frac{1}{\epsilon^2} F(u) + \frac{1}{2} |\nabla u^2| \right] dx. \tag{3}$$

式(3)中: $F(u) = \frac{1}{4}(u^2-1)^2$.

能量泛函 $E(u)$ 关于时间 t 求 Caputo 型分数阶导数为

$$\begin{aligned} {}^C_0D_t^\alpha E(u) &= \int_a^b \left[\frac{1}{\epsilon^2} F'(u) {}^C_0D_t^\alpha u(x,t) + \nabla u \cdot \nabla {}^C_0D_t^\alpha u(x,t) \right] dx = \\ &= \int_a^b \left[\frac{1}{\epsilon^2} F'(u) - \Delta u \right] {}^C_0D_t^\alpha u(x,t) dx = - \int_a^b [{}^C_0D_t^\alpha u(x,t)]^2 dx \leq 0. \end{aligned} \tag{4}$$

由此易知, 能量泛函 $E(u)$ 不会随时间的增长而增加.

求解上述方程数值解的方法很多^[8-20]. 文中利用 Laplace 变换法^[20]逼近时间分数阶 Allen-Cahn 方程(1), 并将其转化为整数阶问题. 然后, 对所得到的整数阶 Allen-Cahn 方程, 采用算子分裂法^[14]将方程分裂为线性部分和非线性部分, 并将解算子分别记为 S^A 和 S^B . 上述方程可通过以下格式求解, 即

$$u(t + \Delta t) = S^B \left(\frac{\Delta t}{2} \right) S^A(\Delta t) S^B \left(\frac{\Delta t}{2} \right) u(t) + O((\Delta t)^2). \tag{5}$$

式(5)中: 线性部分利用 C-N 格式求解, 非线性部分解析求解, 从而达到减少计算量的目的, 且得到简单有效的数值格式.

2 分数阶 Allen-Cahn 方程的数值解

2.1 利用 Laplace 变换将分数阶问题转化为整数阶问题

先用 Laplace 变换逼近 Caputo 型分数阶导数, 即

$$L\{{}^C_0D_t^\alpha u(x,t)\} = s^\alpha \hat{u}(x,s) - s^{\alpha-1} u(x,0) = s^\alpha [\hat{u}(x,s) - s^{-1} u(x,0)]. \tag{6}$$

此处的 $\hat{u}(x,s)$ 为 $u(x,t)$ 的 Laplace 变换. 由于 $\alpha \in (0,1)$, 故可将 s^α 线性化为

$$s^\alpha \approx \alpha s^1 + (1-\alpha)s^0 = \alpha s + (1-\alpha). \tag{7}$$

将式(7)代入到式(6), 可得

$$\begin{aligned} L\{{}^C_0D_t^\alpha u(x,t)\} &\approx [\alpha s + (1-\alpha)][\hat{u}(x,s) - s^{-1} u(x,0)] = \\ &\alpha s [\hat{u}(x,s) - s^{-1} u(x,0)] + (1-\alpha)[\hat{u}(x,s) - s^{-1} u(x,0)]. \end{aligned} \tag{8}$$

再利用 Laplace 逆变换, 可得

$${}^C_0D_t^\alpha u(x,t) \approx \alpha \frac{\partial u(x,t)}{\partial t} + (1-\alpha)[u(x,t) - u(x,0)]. \tag{9}$$

从而原分数阶 Allen-Cahn 方程可转化为整数阶方程, 即

$$\frac{\partial u}{\partial t} = \frac{1}{\alpha} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\alpha-1}{\alpha} [u(x,t) - u(x,0)] - \frac{1}{(\epsilon\sqrt{\alpha})^2} u(u^2-1). \tag{10}$$

2.2 算子分裂法求解 Allen-Cahn 方程

取空间节点数为 M , 空间步长为 $h = (b-a)/M$, 空间节点表示为 $x_i = a + ih, i = 0, 1, \dots, M$; 取时间节点数为 N , 时间步长为 $\Delta t = T/N$, 时间节点表示为 $t_k = k\Delta t, k = 0, 1, \dots, N$, 且记 $u_i^k = u(x_i, t_k)$. 下面给出方程(10)的求解策略.

利用算子分裂将 Allen-Cahn 方程分解为热传导方程和非线性方程, 即有

$$S^A: \frac{\partial u}{\partial t} = \frac{1}{\alpha} \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\alpha-1}{\alpha} [u(x, t) - u(x, 0)], \quad (11)$$

$$S^B: \frac{\partial u}{\partial t} + \frac{1}{(\epsilon \sqrt{\alpha})^2} u(u^2 - 1) = 0. \quad (12)$$

在前半个时间步长 $t \in [t_k, t_{k+\frac{1}{2}}]$ 内, 通过算子 S^B 求解; 然后, 在一个完整的时间步长 $t \in [t_k, t_{k+1}]$ 内, 利用算子 S^A 求解, 且 $u^k = u^{k+\frac{1}{2}}$; 最后, 在后半个时间步长 $t \in [t_{k+\frac{1}{2}}, t_{k+1}]$ 内, 再次利用算子 S^B 求解, 且 $u^{k+\frac{1}{2}} = u^{k+1}$. 即

$$u(t + \Delta t) = S^B\left(\frac{\Delta t}{2}\right) S^A(\Delta t) S^B\left(\frac{\Delta t}{2}\right) u(t). \quad (13)$$

非线性方程解析求解, 其数值格式为

$$u_i^{k+1} = \frac{u_i^k}{\sqrt{\exp(-2\Delta t/(\alpha \epsilon^2)) + (u_i^k)^2 (1 - \exp(-2\Delta t/(\alpha \epsilon^2)))}}. \quad (14)$$

引入二阶中心差分算子, 则有

$$\delta_x^2 u_i^k = \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2}. \quad (15)$$

热传导方程的 C-N 格式为

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = \frac{1}{2\alpha} [\delta_x^2 u_i^k + \delta_x^2 u_i^{k+1}] + \left(1 - \frac{1}{\alpha}\right) \left(\frac{u_i^k + u_i^{k+1}}{2} - u_i^0\right). \quad (16)$$

将式(15)代入式(16)中, 进一步化简, 可得

$$\begin{aligned} & -\frac{\Delta t}{2\alpha h^2} u_{i-1}^{k+1} + \left[1 + \frac{\Delta t}{\alpha h^2} + \frac{(1-\alpha)\Delta t}{2\alpha}\right] u_i^{k+1} - \frac{\Delta t}{2\alpha h^2} u_{i+1}^{k+1} = \\ & \frac{\Delta t}{2\alpha h^2} u_{i-1}^k + \left[1 - \frac{\Delta t}{\alpha h^2} + \frac{(\alpha-1)\Delta t}{2\alpha}\right] u_i^k + \frac{\Delta t}{2\alpha h^2} u_{i+1}^k + \frac{1-\alpha}{\alpha} \Delta t u_i^0. \end{aligned} \quad (17)$$

结合式(13), (14)和(17), 可得到求解问题(10)的二阶差分格式为

$$\left. \begin{aligned} u_i^{k+1, (1)} &= \frac{u_i^k}{\sqrt{\exp(-\Delta t/(\alpha \epsilon^2)) + (u_i^k)^2 (1 - \exp(-\Delta t/(\alpha \epsilon^2)))}}, \\ & -\frac{\Delta t}{2\alpha h^2} u_{i-1}^{k+1, (2)} + \left[1 + \frac{\Delta t}{\alpha h^2} + \frac{(1-\alpha)\Delta t}{2\alpha}\right] u_i^{k+1, (2)} - \frac{\Delta t}{2\alpha h^2} u_{i+1}^{k+1, (2)} = \\ & \frac{\Delta t}{2\alpha h^2} u_{i-1}^{k+1, (1)} + \left[1 - \frac{\Delta t}{\alpha h^2} + \frac{(\alpha-1)\Delta t}{2\alpha}\right] u_i^{k+1, (1)} + \frac{\Delta t}{2\alpha h^2} u_{i+1}^{k+1, (1)} + \frac{1-\alpha}{\alpha} \Delta t u_i^0, \\ u_i^{k+1} &= \frac{u_i^{k+1, (2)}}{\sqrt{\exp(-\Delta t/(\alpha \epsilon^2)) + (u_i^{k+1, (2)})^2 (1 - \exp(-\Delta t/(\alpha \epsilon^2)))}}. \end{aligned} \right\} \quad (18)$$

注 1 Laplace 变换同样可以用来逼近 Riemann-Liouville 型分数阶导数, 而且 $p(>0)$ 阶 Riemann-Liouville 分数阶导数的 Laplace 变换为

$$L\{ {}_0 D_t^p u(x); s \} = s^p \hat{u}(x) - \sum_{k=0}^{n-1} s^k [{}_0 D_t^{p-k-1} u(x)]_{t=0}, \quad n-1 \leq p < n. \quad (19)$$

3 数值算例

通过数值算例, 验证数值格式的有效性和精确性. 为方便分析, 对如下符号进行解释

$$E(M, N) = \max_{1 \leq i \leq M-1} \frac{|u(x_i, T) - u_i^N|}{|u(x_i, T)|}. \quad (20)$$

式(20)中: M 为空间剖分数; N 为时间剖分数; $u(x_i, t_k)$, u_i^k 分别表示精确解与数值解.

3.1 算例一

为验证时间数值计算的精度, 选取具有充分正则性的精确解的方程作为测试实例. 考虑如下 Allen-Cahn 方程, 有

$$\left. \begin{aligned} {}^c D_t^\alpha u(x,t) + \frac{1}{\epsilon^2} u(u^2 - 1) &= \frac{\partial^2 u}{\partial x^2} + l(x,t), \\ u(x,t) &= (t^3 + 1) \sin(1.5\pi x). \end{aligned} \right\} \tag{21}$$

在式(21)右端添加 $l(x,t)$, 则

$$\begin{aligned} l(x,t) &= \frac{\sin(1.5\pi x)}{\Gamma(1-\alpha)} \left(\frac{3}{3-\alpha} - \frac{6}{2-\alpha} + \frac{3}{1-\alpha} \right) t^{3-\alpha} + \\ &\frac{1}{\epsilon^2} [(t^3 + 1) \sin(1.5\pi x)]^3 - \left(\frac{1}{\epsilon^2} - \frac{9}{4} \pi^2 \right) (t^3 + 1) \sin(1.5\pi x) \end{aligned} \tag{22}$$

是为了满足给定的方程及其精确解. 其区域的取值范围是 $[0,1] \times [0,1]$, $\epsilon=0.5$.

取网格剖分 $M=20, N=3\,000$, 给出 α 为 0.5 时的数值解和误差(e)图像, 分别如图 1, 2 所示.

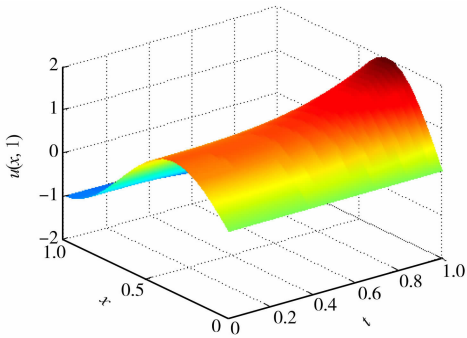


图 1 算例一的数值解图像($\alpha=0.5$)

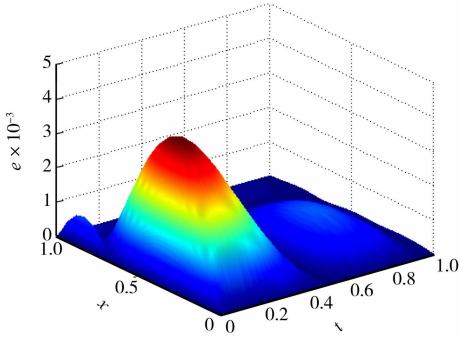


图 2 算例一的误差图像($\alpha=0.5$)

Fig. 1 Numerical solution image of example 1 ($\alpha=0.5$) Fig. 2 Error image of example 1 ($\alpha=0.5$)

分别计算不同剖分、不同 ϵ 和不同 α 时的最大相对误差, 结果如表 1 所示.

表 1 不同 ϵ 时的最大相对误差

Tab. 1 Maximum relative error at different ϵ

| ϵ | E | α | | | | |
|------------|------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| | | 0.2 | 0.5 | 0.7 | 0.9 | 1.0 |
| 0.3 | $E(50, 2\,000)$ | $3.033\,7 \times 10^{-3}$ | $4.085\,7 \times 10^{-3}$ | $3.818\,6 \times 10^{-3}$ | $1.077\,1 \times 10^{-3}$ | $1.267\,8 \times 10^{-3}$ |
| | $E(50, 2\,500)$ | $1.973\,0 \times 10^{-3}$ | $4.382\,6 \times 10^{-3}$ | $3.970\,0 \times 10^{-3}$ | $1.168\,8 \times 10^{-3}$ | $1.193\,4 \times 10^{-3}$ |
| | $E(50, 3\,000)$ | $1.393\,4 \times 10^{-3}$ | $4.543\,9 \times 10^{-3}$ | $4.052\,2 \times 10^{-3}$ | $1.218\,6 \times 10^{-3}$ | $1.152\,9 \times 10^{-3}$ |
| | $E(100, 3\,000)$ | $1.459\,6 \times 10^{-3}$ | $7.110\,4 \times 10^{-3}$ | $6.696\,2 \times 10^{-3}$ | $2.846\,9 \times 10^{-3}$ | $5.373\,9 \times 10^{-4}$ |
| | $E(100, 4\,000)$ | $1.927\,6 \times 10^{-3}$ | $7.395\,3 \times 10^{-3}$ | $6.840\,7 \times 10^{-3}$ | $2.934\,0 \times 10^{-3}$ | $4.669\,3 \times 10^{-4}$ |
| 0.5 | $E(50, 2\,000)$ | $7.972\,4 \times 10^{-3}$ | $1.752\,2 \times 10^{-2}$ | $1.533\,8 \times 10^{-2}$ | $4.947\,4 \times 10^{-3}$ | $3.742\,3 \times 10^{-3}$ |
| | $E(50, 2\,500)$ | $8.300\,7 \times 10^{-3}$ | $1.757\,7 \times 10^{-2}$ | $1.536\,7 \times 10^{-2}$ | $4.965\,1 \times 10^{-3}$ | $3.727\,7 \times 10^{-3}$ |
| | $E(50, 3\,000)$ | $8.479\,0 \times 10^{-3}$ | $1.760\,6 \times 10^{-2}$ | $1.538\,2 \times 10^{-2}$ | $4.974\,7 \times 10^{-3}$ | $3.719\,8 \times 10^{-3}$ |
| | $E(100, 3\,000)$ | $1.874\,0 \times 10^{-2}$ | $3.516\,7 \times 10^{-2}$ | $3.168\,7 \times 10^{-2}$ | $1.366\,2 \times 10^{-2}$ | $3.727\,7 \times 10^{-3}$ |
| | $E(100, 4\,000)$ | $1.907\,4 \times 10^{-2}$ | $3.522\,2 \times 10^{-2}$ | $3.171\,5 \times 10^{-2}$ | $1.368\,0 \times 10^{-2}$ | $1.644\,0 \times 10^{-3}$ |
| 0.7 | $E(50, 2\,000)$ | $1.567\,5 \times 10^{-2}$ | $3.148\,2 \times 10^{-2}$ | $2.741\,9 \times 10^{-2}$ | $8.946\,6 \times 10^{-3}$ | $6.371\,6 \times 10^{-3}$ |
| | $E(50, 2\,500)$ | $1.575\,8 \times 10^{-2}$ | $3.149\,7 \times 10^{-2}$ | $2.742\,7 \times 10^{-2}$ | $8.951\,7 \times 10^{-3}$ | $6.367\,3 \times 10^{-3}$ |
| | $E(50, 3\,000)$ | $1.580\,3 \times 10^{-2}$ | $3.150\,5 \times 10^{-2}$ | $2.743\,2 \times 10^{-2}$ | $8.954\,5 \times 10^{-3}$ | $6.365\,0 \times 10^{-3}$ |
| | $E(100, 3\,000)$ | $3.638\,9 \times 10^{-2}$ | $6.559\,3 \times 10^{-2}$ | $5.864\,3 \times 10^{-2}$ | $2.514\,8 \times 10^{-2}$ | $2.912\,1 \times 10^{-3}$ |
| | $E(100, 4\,000)$ | $3.647\,4 \times 10^{-2}$ | $6.560\,8 \times 10^{-2}$ | $5.865\,1 \times 10^{-2}$ | $2.515\,3 \times 10^{-2}$ | $2.907\,9 \times 10^{-3}$ |

由图 1, 2 可知: 数值解逼近于精确解, 具有较高的精度. 由表 1 可知: 数值解在不同剖分、不同 ϵ 及不同 α 时均满足精度要求, α 越接近 1, ϵ 越小且网格剖分越细密, 数值解精度越高.

3.2 算例二

考虑如下初值问题

$$u(x, 0) = \epsilon \cdot \sin(1.5\pi x), \quad x \in [-1, 1].$$

取 Dirichlet 边界条件, 左边界 $u_0=1$, 右边界 $u_M=-1, t \in [0, T]$. 定义离散能量函数为

$$E^h(u^k) = \frac{h}{4\epsilon^2} \sum_{i=0}^M [(u_i^k)^2 - 1]^2 + \frac{h}{2} \sum_{i=1}^{M-1} \left[\frac{u_{i+1}^k - u_{i-1}^k}{2h} \right]^2.$$

(23)

具体求解参数为 $M=20, N=1\ 000, T=2, \epsilon=0.1$, 并分别取 $\alpha=0.2, 0.5, 0.9$, 得到不同 α 的数值解和能量变化图像, 分别如图 3~8 所示.

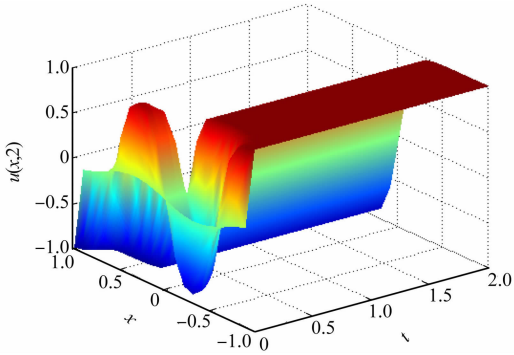


图 3 算例二的数值解图像($\alpha=0.2$)

Fig. 3 Numerical solution image of example 2 ($\alpha=0.2$)

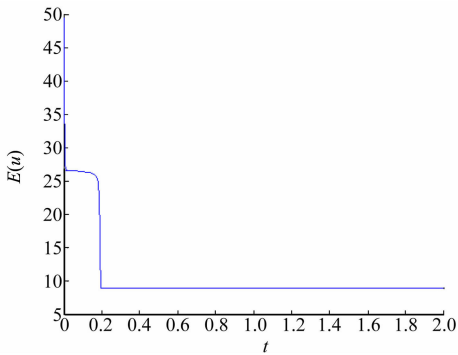


图 4 算例二的能量变化图像($\alpha=0.2$)

Fig. 4 Energy change image of example 2 ($\alpha=0.2$)

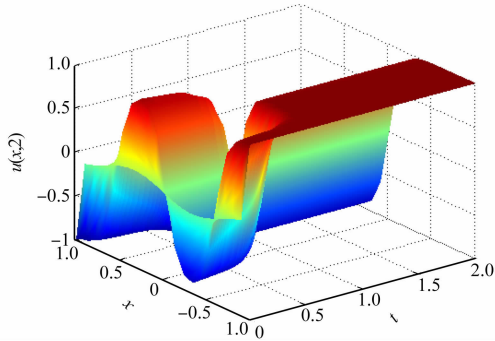


图 5 算例二的数值解图像($\alpha=0.5$)

Fig. 5 Numerical solution image of example 2 ($\alpha=0.5$)

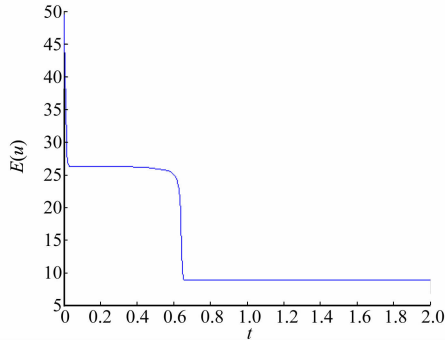


图 6 算例二的能量变化图像($\alpha=0.5$)

Fig. 6 Energy change image of example 2 ($\alpha=0.5$)

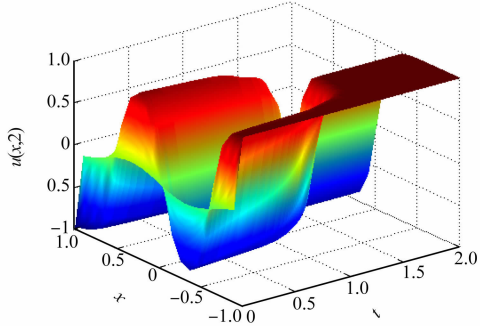


图 7 算例二的数值解图像($\alpha=0.9$)

Fig. 7 Numerical solution image of example 2 ($\alpha=0.9$)

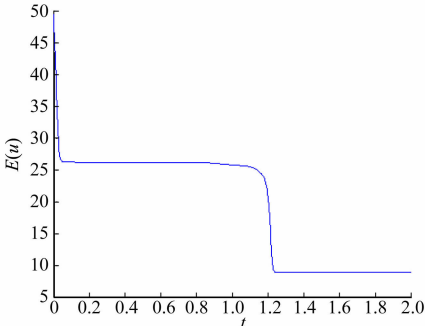


图 8 算例二的能量变化图像($\alpha=0.9$)

Fig. 8 Energy change image of example 2 ($\alpha=0.9$)

由图 3~8 可知:能量函数 $E(u)$ 随着时间 t 的增大而减小, 即能量泛函 $E(u)$ 满足能量递减. 此外, 时间分数阶 Allen-Cahn 方程的能量耗散受分数阶 α 的影响, α 越小, 能量衰减越快.

4 结束语

利用 Laplace 变换, 将时间分数阶 Allen-Cahn 方程转化为整数阶 Allen-Cahn 方程; 然后, 再利用算子分裂法得到能量稳定的二阶差分格式; 最后, 通过数值算例验证了格式的有效性.

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