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椭圆界面问题的高阶差分格式

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摘要：构造混合边界条件下椭圆界面问题的一个高阶数值格式. 在求解区域内部及界面处采用四阶逼近，边界处采用三阶数值格式，得到一个整体四阶精度的求解格式. 数值实验证明了格式的高精度及有效性.

关键词：椭圆界面问题；混合边界；四阶 Padé 逼近；高阶数值

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High-Order Finite Difference Scheme for Elliptic Interface Problem

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Abstract: In this paper, we propose a high-order finite difference scheme for elliptic interface problems with mixed boundary conditions. The fourth-order approximation is adopted in the solution area and the interface, while a third-order numerical scheme is adopted on the boundary, we obtain a solution scheme with global fourth order accuracy. Numerical experiments are given to illustrate the high accuracy and effectiveness of our scheme.

Keywords: elliptical interface problem; mixed boundary; fourth-order Padé approximation; higher-order values

1 预备知识

椭圆界面问题是一类在流体动力学、分子生物学[1]、电磁学和材料科学中广泛存在的问题. 考虑如下椭圆界面问题，即

$$\left.\begin{aligned} &k_1\Delta u = f_1, \qquad x\in[0,d^-],\\ &k_2\Delta u = f_2, \qquad x\in[d^+,1],\\ &u(d^+,y)-u(d^-,y)=a(y),\\ &k_2u_x(d^+,y)-k_1u_x(d^-,y)=b(y).\end{aligned}\right\} \tag{1}$$

式(1)中：$(x,y)\in[0,1]^2$，界面区域 $\Gamma=\{(x,y)\mid x=d,y\in[0,1]\}$，边界条件为

$$\left.\begin{aligned} &\frac{\partial u(0,y)}{\partial x}=\frac{\partial u(1,y)}{\partial x}=0, \qquad 0\leqslant y\leqslant 1,\\ &u(x,0)=u(x,1)=0, \qquad 0\leqslant x\leqslant 1.\end{aligned}\right\} \tag{2}$$

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显然，问题(1)的解在界面区域 Γ 上是不光滑的，甚至是不连续的. 传统的数值方法已经不适用于椭圆界面问题. 问题(1)的数值工作除了佩斯金的浸入边界法[2-6]之外，还有很多学者进行相关研究[7-11]. 此外，有限元在椭圆界面问题中也得到了广泛应用[12-14]. 然而，以上方法都为二阶精度. 本文利用差分法构建椭圆界面问题的一个高级数值格式. 在求解区域内部及界面处采用四阶逼近，在边界处应用三阶数值格式，最终得到一个具有四阶精度的数值格式[15].

2 基本引理

首先给出两个引理.

引理 1 假设 $g(x)\in C^6[0,L]$，将区间 $[0,L]$ M 等分得到一组点 x_i，$i=0,1,2,\cdots,M$，则有

$$\frac{5}{12}g_{xx}(x_0)+\frac{1}{12}g_{xx}(x_1)=\frac{1}{h^2}[g(x_1)-g(x_0)-hg_x(x_0)]-\frac{1}{12}hg_{xxx}(x_0)+O(h^3),\tag{3}$$

$$\frac{1}{12}g_{xx}(x_{M-1})+\frac{5}{12}g_{xx}(x_M)=\frac{1}{h^2}[g_x(x_{M-1})-g(x_M)+hg_x(x_M)]+\frac{1}{12}hg_{xxx}(x_M)+O(h^3).\tag{4}$$

引理 2[15] 假设 $g(x)\in C^6[0,L]$，将区间 $[0,L]$ M 等分得到一组点 x_i，$i=0,1,2,\cdots,M$. 则对任意的 $1\leqslant I\leqslant M-1$，有

$$-\frac{1}{3}g_{xx}(x_{m-1})+\frac{8}{3}g_{xx}(x_m)+\frac{35}{3}g_{xx}(x_{I^-})=\frac{1}{h^2}[32g(x_m)-g(x_{m-1})-$$
$$31g(x_{I^-})+30hg_x(x_{I^-})]+2hg_{xxx}(x_{I^-})+O(h^4),\tag{5}$$

$$\frac{35}{3}g_{xx}(x_{I^+})+\frac{8}{3}g_{xx}(x_{m+1})-\frac{1}{3}g_{xx}(x_{m+2})=\frac{1}{h^2}[32g(x_{m+1})-g(x_{m+2})-$$
$$31g(x_{I^+})-30hg_x(x_{I^+})]-2hg_{xxx}(x_{I^+})+O(h^4).\tag{6}$$

3 数值离散

令 x 方向取空间节点数 M，步长 $h_x=1/M$，则 x 方向空间节点为 $x_i=ih_x$，$i=0,1,\cdots m,I,m+1,\cdots,M$. 其中，$d=Ih_x$；$y$ 方向空间节点数为 N，步长 $h_y=1/N$，则 y 方向空间节点为 $y_j=jh_y$，$j=0,1,\cdots,N$. 记 $u_i^j=u(x_i,y_j)$，并定义两个二阶中心差分算子，即

$$\delta_x^2u_i^j=\frac{u_{i-1}^j-2u_i^j+u_{i+1}^j}{h_x^2},\qquad \delta_y^2u_i^j=\frac{u_i^{j-1}-2u_i^{j-1}+u_i^{j+1}}{h_y^2}.\tag{7}$$

由此得到相应四阶 Padé 逼近[16-17]，即

$$\frac{\partial^2u(x_i,y_j)}{\partial y^2}=\frac{\delta_y^2u(x_i,y_j)}{1+\frac{h_y^2}{12}\frac{\mathrm{d}^2}{\mathrm{d}y^2}}+O(h_y^4),\qquad \frac{\partial^2u(x_i,y_j)}{\partial x^2}=\frac{\delta_x^2u(x_i,y_j)}{1+\frac{h_x^2}{12}\frac{\mathrm{d}^2}{\mathrm{d}x^2}}+O(h_x^4).\tag{8}$$

为方便表达，记 $L_x=1+\frac{h_x^2}{12}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$，$L_y=1+\frac{h_y^2}{12}\frac{\mathrm{d}^2}{\mathrm{d}y^2}$. 下面具体离散问题(1).

当 $i=1$ 时，由引理 1 可知

$$\frac{5}{12}\frac{\partial^2u}{\partial x^2}(x_0,y_j)+\frac{1}{12}\frac{\partial^2u}{\partial x^2}(x_1,y_j)=\frac{1}{h_x^2}[u(x_1,y_j)-u(x_0,y_j)-$$
$$h_xu_x(x_0,y_j)]-\frac{h_x}{12}\frac{\partial^3u}{\partial x^3}(x_0,y_j)+O(h_x^3).\tag{9}$$

由问题(1)的第 1 式有 $\frac{\partial^2u}{\partial x^2}=\frac{1}{k_1}f_1-\frac{\partial^2u}{\partial y^2}$，将其代入式(9)中，有

$$\left(-\frac{1}{h_x^2}+\frac{5}{12}\frac{\partial^2}{\partial y^2}\right)u(x_0,y_j)+\left(\frac{1}{h_x^2}+\frac{1}{12}\frac{\partial^2}{\partial y^2}\right)u(x_1,y_j)=\frac{5}{12k_1}f_1(x_0,y_j)+$$
$$\frac{1}{12k_1}f_1(x_1,y_j)+\frac{h_x}{12k_1}f_{1,x}(x_0,y_j)-\left(-\frac{1}{h_x}+\frac{h_x}{12}\frac{\partial^2}{\partial y^2}\right)u_x(x_0,y_j)+O(h_x^3).\tag{10}$$

两边同乘以 L_y，则有

$$\left(-\frac{L_y}{h_x^2}+\frac{5}{12}\delta_y^2\right)u(x_0,y_j)+\left(\frac{L_y}{h_x^2}+\frac{1}{12}\delta_y^2\right)u(x_1,y_j)=\frac{5L_y}{12k_1}f_1(x_0,y_j)+$$

$$\frac{L_y}{12k_1}f_1(x_1,y_j)+\frac{h_xL_y}{12k_1}f_{1,x}(x_0,y_j)\frac{L_y}{h_x}+\frac{h_x}{12}\delta_y^2u_x(x_0,y_j)+O(h_x^3). \tag{11}$$

用数值解代替真解,省略误差项,可得

$$\left(-\frac{L_y}{h_x^2}+\frac{5}{12}\delta_y^2\right)u_0^j+\left(\frac{L_y}{h_x^2}+\frac{1}{12}\delta_y^2\right)u_1^j=\frac{5L_y}{12k_1}f_{1,0}^j+\frac{L_y}{12k_1}f_{1,1}^j+$$

$$\frac{h_xL_y}{12k_1}(f_{1,x})_0^j-\left(-\frac{L_y}{h_x}+\frac{h_x}{12}\delta_y^2\right)(u_x)_0^j. \tag{12}$$

当 $1<i<I^-$ 时,对问题(1)的第1式两边同时乘以 L_xL_y,可得

$$k_1(L_x\delta_y^2+L_y\delta_x^2)u_i^j=L_xL_yf_1. \tag{13}$$

特别的,当 $i=m$ 时,根据式(13)可得

$$\left(\frac{1}{12}\delta_y^2+\frac{1}{h_x^2}L_y\right)u_{m-1}^j+\left(\frac{5}{6}\delta_y^2-\frac{2}{h_x^2}L_y\right)u_m^j+\left(\frac{1}{12}\delta_y^2+\frac{1}{h_x^2}L_y\right)u_{I^-}^j=\frac{1}{k_1}L_xL_yf_{1,m}^j. \tag{14}$$

当 $i=I^-$ 时,根据引理2有

$$\frac{1}{h_x^2}(u_{m-1}^j-32u_m^j+31u_{I^-}^j-30h_x(u_x)_{I^-}^j)=-\frac{1}{3}\left[35\frac{\partial^2u}{\partial x^2}(x_{I^-},y_j)+\right.$$

$$\left.8\frac{\partial^2u}{\partial x^2}(x_m,y_j)-\frac{\partial^2u}{\partial x^2}(x_{m-1},y_j)\right]+2h_x\frac{\partial^3u}{\partial x^3}(x_{I^-},y_j)+O(h_x^4). \tag{15}$$

由问题(1)的第1式有 $\frac{\partial^2u}{\partial x^2}=\frac{1}{k_1}f_1-\frac{\partial^2u}{\partial y^2}$,将其代入式(15)中,可得

$$\frac{1}{h_x^2}(u_{m-1}^j-32u_m^j+31u_{I^-}^j-30h_x(u_x)_{I^-}^j)=-\frac{1}{3k_1}[35f_{1,I^-}^j+8f_{1,m}^j-f_{1,m-1}^j]+2\frac{h_x}{k_1}(f_{1,x})_{I^-}^j+$$

$$\frac{1}{3}\left[35\frac{\partial^2u}{\partial y^2}(x_{I^-},y_j)+8\frac{\partial^2u}{\partial y^2}(x_m,y_j)-\frac{\partial^2u}{\partial y^2}(x_{m-1},y_j)\right]-2h_x\frac{\partial^2u_x}{\partial y^2}(x_{I^-},y_j)+O(h_x^4). \tag{16}$$

令 $c_1=-\frac{1}{3k_1}[35f_{1,I^-}^j+8f_{1,m}^j-f_{1,m-1}^j]+2\frac{h_x}{k_1}(f_{1,x})_{I^-}^j$,对式(16)两边同时乘以 k_1L_y,可得

$$\frac{k_1L_y}{h_x^2}(u_{m-1}^j-32u_m^j+31u_{I^-}^j-30h_x(u_x)_{I^-}^j)=\frac{k_1}{3}[35\delta_y^2u_{I^-}^j+$$

$$8\delta_y^2u_m^j-\delta_y^2u_{m-1}^j]-2k_1h_x\delta_y^2(u_x)_{I^-}^j+k_1L_yc_1+O(h_x^4+h_y^4). \tag{17}$$

经整理可得

$$k_1\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m-1}^j-k_1\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_m^j+k_1\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)u_{I^-}^j=$$

$$k_1L_yc_1+k_1h_x\left(30\frac{L_y}{h_x^2}-2\delta_y^2\right)(u_x)_{I^-}^j+O(h_x^4+h_y^4). \tag{18}$$

当 $i=I^+$ 时,利用引理2有

$$\frac{1}{h_x^2}(u_{m+2}^j-32u_{m+1}^j+31u_{I^-}^j+30h_x(u_x)_{I^+}^j)=-\frac{1}{3}\left[35\frac{\partial^2u}{\partial x^2}(x_{I^+},y_j)+\right.$$

$$\left.8\frac{\partial^2u}{\partial x^2}(x_{m+1},y_j)-\frac{\partial^2u}{\partial x^2}(x_{m+2},y_j)\right]-2h_x\frac{\partial^3u}{\partial x^3}(x_{I^+},y_j)+O(h_x^4). \tag{19}$$

由问题(1)的第2式有 $\frac{\partial^2u}{\partial x^2}=\frac{1}{k_2}f_2-\frac{\partial^2u}{\partial y^2}$,将其代入式(19)中,可得

$$\frac{1}{h_x^2}(u_{m+2}^j-32u_{m+1}^j+31u_{I^+}^j+30h_x(u_x)_{I^+}^j)=-\frac{1}{3k_2}[35f_{2,I^+}^j+8f_{2,m+1}^j-f_{1,m+2}^j]-$$

$$2\frac{h_x}{k_2}(f_{2,x})_{I^+}^j+\frac{1}{3}\left[35\frac{\partial^2u}{\partial y^2}(x_{I^+},y_j)+8\frac{\partial^2u}{\partial y^2}(x_{m+1},y_j)-\frac{\partial^2u}{\partial y^2}(x_{m+2},y_j)\right]+$$

$$2h_x\frac{\partial^2u_x}{\partial y^2}(x_{I^+},y_j)+O(h_x^4). \tag{20}$$

令 $c_2=-\dfrac{1}{3k_2}[35f_{2,I^+}^j+8f_{2,m+1}^j-f_{2,m+2}^j]+2\dfrac{h_x}{k_1}(f_{2,x})_{I^+}^j$，对方程两边同时乘以 k_2L_y，可得

$$k_2\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m+2}^j-k_2\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_{m+1}^j+k_2\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)u_{I^+}^j=$$
$$k_2L_yc_2+k_2h_x\left(-30\frac{L_y}{h_x^2}+2\delta_y^2\right)(u_x)_{I^+}^j+O(h_x^4+h_y^4).\tag{21}$$

通过将式(21)和式(18)相加，可得

$$k_2\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m+2}^j-k_2\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_{m+1}^j+\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)(k_2u_{I^+}^j+k_1u_{I^-}^j)+$$
$$k_1\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m-1}^j-k_1\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_m^j=k_1L_yc_1+k_2L_yc_2+$$
$$h_x\left(-30\frac{L_y}{h_x^2}+2h_x\delta_y^2\right)(k_2(u_x)_{I^+}^j-k_1(u_x)_{I^-}^j)+O(h_x^4+h_y^4).\tag{22}$$

由问题(1)的第 3 式可知 $u(d^+,y)=u(d^-,y)+a(y)$，将其和问题(1)的第 4 式代入式(22)中，可得

$$k_2\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m+2}^j-k_2\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_{m+1}^j+\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)(k_2+k_1)u_{I^-}^j+$$
$$k_1\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m-1}^j-k_1\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_m^j=k_1L_yc_1+k_2L_yc_2+$$
$$\left(-30\frac{L_y}{h_x^2}+2\delta_y^2\right)h_xb^j-k_2\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)a^j+O(h_x^4+h_y^4).\tag{23}$$

令 $c_3=k_1L_yc_1+k_2L_yc_2+\left(-30\dfrac{L_y}{h_x^2}+2\delta_y^2\right)h_xb^j-k_2\left(31\dfrac{L_y}{h_x^2}-\dfrac{35}{3}\delta_y^2\right)a^j$，省略误差项，可得

$$k_2\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m+2}^j-k_2\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_{m+1}^j+\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)(k_2+k_1)u_{I^-}^j+$$
$$k_1\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m-1}^j-k_1\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_m^j=c_3.\tag{24}$$

当 $I^-<i<M$ 时，对问题(1)的第 2 式两边同时乘以 L_xL_y，则有

$$k_2(L_x\delta_y^2+L_y\delta_x^2)u_i^j=L_xL_yf_{2,i}^j.\tag{25}$$

当 $i=M$ 时，由引理 1 可知

$$\frac{1}{12}\frac{\partial^2u}{\partial x^2}(x_{M-1},y_j)+\frac{5}{12}\frac{\partial^2u}{\partial x^2}(x_M,y_j)=\frac{1}{h_x^2}[u(x_{M-1},y_j)-$$
$$u(x_M,y_j)+h_xu_x(x_M,y_j)]+\frac{h_x}{12}\frac{\partial^3u}{\partial x^3}(x_M,y_j)+O(h_x^3).\tag{26}$$

由问题(1)的第 2 式有$\dfrac{\partial^2u}{\partial x^2}=\dfrac{1}{k_2}f_2-\dfrac{\partial^2u}{\partial y^2}$，将其代入式(26)中，有

$$\left(\frac{1}{h_x^2}+\frac{1}{12}\frac{\partial^2}{\partial y^2}\right)u(x_{M-1},y_j)+\left(-\frac{1}{h_x^2}+\frac{5}{12}\frac{\partial^2}{\partial y^2}\right)u(x_M,y_j)+\left(\frac{1}{h_x}-\frac{h_x}{12}\frac{\partial^2}{\partial y^2}\right)u_x(x_M,y_j)=$$
$$\frac{1}{12k_2}f_2(x_{M-1},y_j)+\frac{5}{12k_2}f_2(x_M,y_j)-\frac{h_x}{12k_2}f_{2,x}(x_M,y_j)+O(h_x^3).\tag{27}$$

两边同乘以 L_y，则有

$$\left(\frac{L_y}{h_x^2}+\frac{1}{12}\delta_y^2\right)u(x_{M-1},y_j)+\left(-\frac{L_y}{h_x^2}+\frac{5}{12}\delta_y^2\right)u(x_M,y_j)=\frac{L_y}{12k_2}f_2(x_{M-1},y_j)+$$
$$\frac{5L_y}{12k_2}f_2(x_M,y_j)-\frac{h_xL_y}{12k_2}f_{2,x}(x_M,y_j)-\left(\frac{L_y}{h_x}-\frac{h_x}{12}\delta_y^2\right)u_x(x_M,y_j)+O(h_x^3).\tag{28}$$

用数值解代替精确解并省略误差项，可得到

$$\left(\frac{L_y}{h_x^2}+\frac{1}{12}\delta_y^2\right)u_{M-1}^j+\left(-\frac{L_y}{h_x^2}+\frac{5}{12}\delta_y^2\right)u_M^j=\frac{L_y}{12k_2}f_{2,M-1}^j+$$
$$\frac{5L_y}{12k_2}f_{2,M}^j-\frac{h_xL_y}{12k_2}(f_{2,x})_M^j-\left(\frac{L_y}{h_x}-\frac{h_x}{12}\delta_y^2\right)(u_x)_M^j+O(h_x^3).\tag{29}$$

整理得到的格式

$$\left.\begin{aligned}
&\left(-\frac{L_y}{h_x^2}+\frac{5}{12}\delta_y^2\right)u_0^j+\left(\frac{L_y}{h_x^2}+\frac{1}{12}\delta_y^2\right)u_1^j=\frac{5L_y}{12k_1}f_{1,0}^j+\frac{L_y}{12k_1}f_{1,1}^j+\\
&\frac{h_xL_y}{12k_1}(f_{1,x})_0^j+\left(\frac{L_y}{h_x}-\frac{h_x}{12}\delta_y^2\right)(u_x)_0^j,\qquad i=1,\\
&k_1(L_x\delta_y^2+L_y\delta_x^2)u_i^j=L_xL_yf_1,\qquad 1<i\leqslant m,\\
&k_2\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m+2}^j-k_2\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_{m+1}^j+\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)(k_2+k_1)u_{I^-}^j+\\
&k_1\left(\frac{L_y}{h_x^2}+\frac{1}{3}\delta_y^2\right)u_{m-1}^j-k_1\left(32\frac{L_y}{h_x^2}+\frac{8}{3}\delta_y^2\right)u_m^j=c_3,\qquad i=I^-,\\
&u_{I^+}^j=u_{I^-}^j+a^j,\qquad i=I^+,\\
&k_2(L_x\delta_y^2+L_y\delta_x^2)u_i^j=L_xL_yf_2,\qquad m+1\leqslant i<M,\\
&\left(\frac{L_y}{h_x^2}+\frac{1}{12}\delta_y^2\right)u_{M-1}^j+\left(-\frac{L_y}{h_x^2}+\frac{5}{12}\delta_y^2\right)u_M^j=\frac{L_y}{12k_2}f_{2,M-1}^j+\frac{5L_y}{12k_2}f_{2,M}^j-\\
&\frac{h_xL_y}{12k_2}(f_{2,x})_M^j-\left(\frac{L_y}{h_x}-\frac{h_x}{12}\delta_y^2\right)(u_x)_M^j,\qquad i=M.
\end{aligned}\right\}\tag{30}$$

式(30)中:$c_3=k_1L_yc_1+k_2L_yc_2+\left(-30\frac{L_y}{h_x^2}+2\delta_y^2\right)h_x b^j-k_2\left(31\frac{L_y}{h_x^2}-\frac{35}{3}\delta_y^2\right)a^j$.

4 数值算例

给出两个数值算例验证格式的有效性.标记真解为 $\hat{u}_i^j=u(x_i,y_j)$,则误差为 $e_i^j=u_i^j-\hat{u}_i^j$.定义两种误差,即

$$\mathrm{Err}_\infty(\tau,h)=\max_{1\leqslant i\leqslant M,1\leqslant j\leqslant M}|e_i^j|,\qquad \mathrm{Err}_2(\tau,h)=h\sqrt{\sum_{i,j=1}^{M}(e_i^j)^2}.$$

4.1 算例1

为验证收敛阶,选取问题

$$\left.\begin{aligned}
&\Delta u=-5\pi^2\cos(\pi x)\sin(2\pi y),\qquad 0\leqslant x<\frac{1}{2},\quad 0\leqslant y\leqslant 1,\\
&\frac{1}{9}\Delta u=-\frac{13}{9}\pi^2\cos(3\pi x)\sin(2\pi y),\qquad \frac{1}{2}<x\leqslant 1,\quad 0\leqslant y\leqslant 1,\\
&u\left(\frac{1}{2}^+,y\right)-u\left(\frac{1}{2}^-,y\right)=0,\qquad 0\leqslant y\leqslant 1,\\
&\frac{1}{9}u_x\left(\frac{1}{2}^+,y\right)-u_x\left(\frac{1}{2}^-,y\right)=\frac{4}{3}\pi\sin(2\pi y),\qquad 0\leqslant y\leqslant 1.
\end{aligned}\right\}\tag{31}$$

其边界条件及初值为

$$\begin{cases}\dfrac{\partial u(0,y)}{\partial x}=\dfrac{\partial u(1,y)}{\partial x}=0,\qquad 0\leqslant y\leqslant 1,\\ u(x,0)=u(x,1)=0,\qquad 0\leqslant x\leqslant 1.\end{cases}$$

其中,真解为 $u_1(x,y)=\cos(\pi x)\sin(2\pi y)$,$u_2(x,y)=\cos(3\pi x)\sin(2\pi y)$.

首先验证收敛阶,x,y 方向剖分 M,N 同时扩大2倍,计算得到表1.从表1可知:随着网格剖分变细,格式的误差越来越小,收敛率也近预期的四阶精度.这说明方法是可行的.图1为算例1的数值图像.从图1可知:格式(30)可以很好地逼近真解.

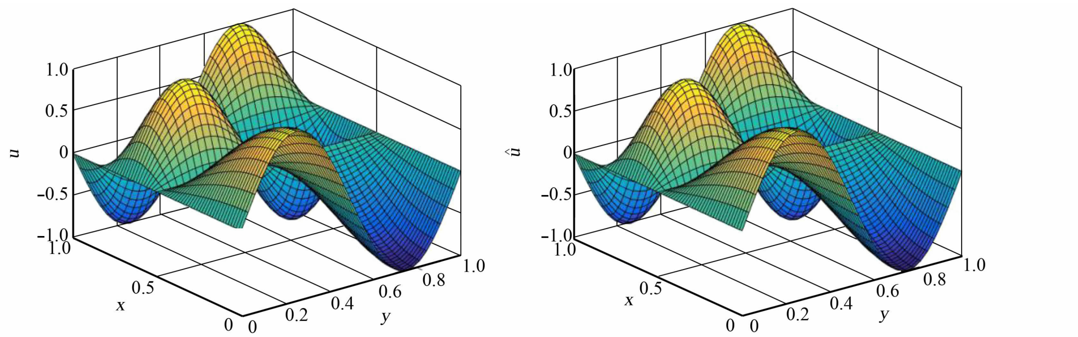
表1 算例1的空间收敛阶

Tab.1 Space convergence order of example 1

M	N	Err_∞	Rate	Err_2	Rate
8	16	1.3882×10^{-3}	—	7.5587×10^{-4}	—
16	32	8.4291×10^{-5}	4.04	4.3494×10^{-5}	4.12
32	64	5.2221×10^{-6}	4.01	2.6295×10^{-6}	4.05
64	128	3.2541×10^{-7}	4.00	1.6191×10^{-7}	4.02
128	256	2.0351×10^{-8}	4.00	1.0048×10^{-8}	4.01

4.2 算例2

为证明格式的一般性,选取如下问题



(a) 数值解　　(b) 真解

图 1　算例 1 的数值图像($M=32,N=64$)

Fig. 1　Numerical image of example 1 ($M=32$, $N=64$)

$$\left.\begin{aligned}
&\frac{1}{2}\Delta u=\frac{1}{4}(4x^4y^3+6x^4y^2+2x^4y-4x^4+10x^3y^3-33x^3y^2+5x^3y+6x^3+\\
&8x^2y^3-6x^2y^2+4x^2y-2x^2-14xy^3+21xy^2-\\
&7xy+2y^3-3y^2+y)\cdot\exp(x+y),\qquad 0\leqslant x<\frac{1}{2},\quad 0\leqslant y\leqslant 1,\\
&\frac{1}{5}\Delta u=\frac{1}{10}(-4x^4y^3-6x^4y^2-2x^4y+4x^4-14x^3y^3+27x^3y^2-7x^3y-2x^3+\\
&8x^2y^3+66x^2y^2+4x^2y-26x^2+20xy^3-144xy^2+10xy+38x+\\
&2y^3+39y^2+y-14)\cdot\exp(x+y),\qquad \frac{1}{2}<x\leqslant 1,\quad 0\leqslant y\leqslant 1,\\
&u\left(\frac{1}{2}^{+},y\right)-u\left(\frac{1}{2}^{-},y\right)=0,\quad 0\leqslant y\leqslant 1,\\
&\frac{1}{5}u_x\left(\frac{1}{2}^{+},y\right)-\frac{1}{2}u_x\left(\frac{1}{2}^{-},y\right)=\frac{11}{160}y(2y^2-3y+1)\cdot\exp(y+0.5),\qquad 0\leqslant y\leqslant 1.
\end{aligned}\right\}\tag{32}$$

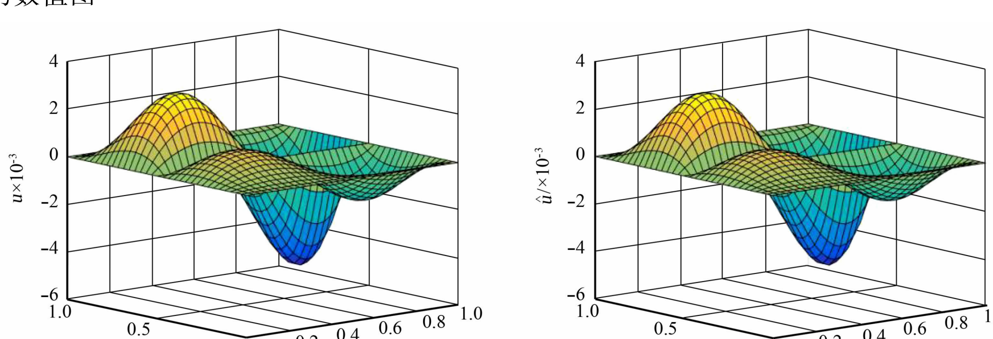
边界条件为$\dfrac{\partial u(0,y)}{\partial x}=\dfrac{\partial u(1,y)}{\partial x}=0$，$0\leqslant y\leqslant 1$，初值为 $u(x,0)=u(x,1)=0$，$0\leqslant x\leqslant 1$. 其中，真解为 $u_1(x,y)=x^2(x-1)(x-1/2)\times y(y-1)(y-1/2)\cdot\exp(x+y)$，$u_2(x,y)=(x-1)^2(2-x)(x-1/2)y(y-1)(y-1/2)\cdot\exp(x+y)$.

首先，验证收敛阶，x，y 方向剖分 M，N 同时扩大 2 倍，计算得到表 2. 从表 2 可知：随着网格剖分变细，收敛率也近预期的四阶精度. 这可说明 k_1 和 k_2 的取值不影响格式的有效性. 图 2 为算例 2 的数值图

表 2　算例 2 的空间收敛阶

Tab. 2　Space convergence order of example 2

M	N	Err_∞	Rate	Err_2	Rate
16	16	$2.026\,8\times10^{-7}$	—	$4.632\,3\times10^{-8}$	—
32	32	$1.311\,4\times10^{-8}$	3.95	$2.748\,1\times10^{-9}$	4.08
64	64	$8.264\,5\times10^{-10}$	3.99	$1.675\,2\times10^{-10}$	4.04
128	128	$5.264\,2\times10^{-11}$	3.99	$1.087\,6\times10^{-11}$	4.02
256	256	$3.296\,4\times10^{-12}$	4.00	$6.779\,7\times10^{-13}$	4.00



(a) 数值解　　(b) 真解

图 2　算例 2 的数值图像($M=32,N=32$)

Fig. 2　Numerical image of example 2 ($M=32$, $N=32$)

像.从图 2 可知:当 u 属于一个很小的量级时,文中的格式依然有效.

5 结束语

提出求解椭圆界面问题的一个高阶数值格式.数值实验验证了格式的有效性,并证明当 u 很小时的格式依然有效.

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