

DOI: 10.11830/ISSN.1000-5013.201707005



四阶常微分方程的 Birkhoff 配点法

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摘要: 提出求解四阶常微分方程的 Birkhoff 配点法. 通过构造满足边界条件的 Birkhoff 插值基函数, 得到具有稳定条件数的代数方程组. 在数值算例中, 通过与一类 Legendre 配点法的数值结果进行比较. 结果表明: Birkhoff 配点法的有效性和高精度.

关键词: 四阶常微分方程; Birkhoff 配点法; Legendre 配点法; 代数方程组

中图分类号: O 241.8

文献标志码: A

文章编号: 1000-5013(2018)02-0306-06

Birkhoff Collocation Method for Fourth-Order Ordinary Differential Equations

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Abstract: The paper presents a Birkhoff collocation method for solving the fourth-order differential equation. The Birkhoff interpolation basis functions satisfying the boundary conditions are constructed, which leads to algebraic equations with stable condition numbers. Numerical results indicate that the Birkhoff collocation method is of high accuracy and effectiveness comparing with a kind of Legendre collocation method.

Keywords: fourth-order ordinary differential equation; Birkhoff collocation method; Legendre collocation method; algebraic equation

四阶微分方程广泛应用于弹性力学和工程物理中, 受到了广大科研工作者的关注^[1-4]. Funaro 等^[5]采用拟谱逼近法求解一维四阶问题. Malek 等^[6]提出用伪谱配点法求解四阶微分方程. Noor 等^[7]给出了求解四阶边值问题的变分迭代法. Quang 等^[3]则将四阶方程转化为相应的两个二阶方程组再进行求解. Zhuang^[4]将 Legendre 谱方法应用于求解夹持杆问题并给出数值结果. Sun 等^[8]则提出一种新的 Galerkin 谱元法求解一维四阶边值问题. 因此, 研究四阶常微分方程的解法具有实际指导意义. 配点法是应用于物理空间的最重要方法之一^[9-13], 也被应用于求解高阶微分方程^[14-16]. Costabile 等^[17]提出用 Birkhoff 插值求解二阶边值问题. 文献[18-19]给出基于 Birkhoff 插值的理论, Wang 等^[20]给出 Birkhoff 配点法求解一阶和二阶边值问题的过程. 由于 Wang 等^[20]并没有给出高阶方程的详细求解过程, 本文将进一步研究用 Birkhoff 配点法求解四阶常微分方程, 并与一类 Legendre 配点法进行对比, 主要考虑方程的数值计算.

1 Birkhoff 配点法

记 $\Delta = (-1, 1)$, 设 α 和 β 为正常数. 考虑如下四阶常微分方程边值问题, 即

$$\left. \begin{aligned} u^{(4)}(x) - \alpha u''(x) + \beta u(x) &= f(x), & x \in \Delta, \\ u(1) &= u^+, & u(-1) = u^-, & u'(1) = u_x^+, & u'(-1) = u_x^-. \end{aligned} \right\} \quad (1)$$

收稿日期: 2017-07-02

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基金项目: 国家自然科学基金资助项目(11501224); 华侨大学中青年教师科研提升资助计划项目(ZQN-PY201)

式(1)中: u^+, u^-, u_x^+, u_x^- 为已知的常数.

1.1 Birkhoff 插值

记 $\{x_j\}_{j=0}^N$ 为区间 $[-1, 1]$ 上 $N+1$ 个互异的插值点, 即 $-1 \leq x_0 < x_1 < \cdots < x_{N-1} < x_N \leq 1$.

设 P_k^* 为次数不超过 k 的代数多项式全体. 对给定的 $k+1$ 个数值 $y_j^m (k \geq N)$, 考虑如下插值问题^[19], 即找 $p_k \in P_k^*$, 使得

$$p_k^{(m)}(x_j) = y_j^m. \tag{2}$$

当 $m=0, 1, L$ 时, m_j 为连续数列, 如当 $m_j=0$ 时, 式(2)为 Lagrange 插值, 当 $m_j=1$ 时, 式(2)为 Hermite 插值; 当 m 为不连续序列时, 式(2)为 Birkhoff 插值. 考虑基于 Legendre-Gauss-Lobatto 点的 Birkhoff 插值, 用 $L_k(x)$ 表示 k 阶 Legendre 多项式, 记 $\{x_j\}_{j=0}^N$ 为 $(1-x^2)L'_N(x)$ 的零点, $\{\omega_j\}_{j=0}^N$ 为相应的权系数. 对 $\forall \varphi \in P_{2N-1}^*$, $\int_{-1}^1 \varphi(x) dx = \sum_{j=0}^N \varphi(x_j) \omega_j$ 等式精确成立^[11].

Legendre 多项式 $\{L_k(x)\}$ 满足如下关系式^[11], 即

$$\int_{-1}^1 L_k(x) L_j(x) dx = \gamma_k \delta_{k,j}, \quad \gamma_k = \frac{2}{2k+1}, \tag{3}$$

$$L_k(x) = \frac{1}{2k+1} (L'_{k+1}(x) - L'_{k-1}(x)), \tag{4}$$

$$L_k(\pm 1) = (\pm 1)^k, \quad L'_k(\pm 1) = \frac{1}{2} (\pm 1)^{k-1} k(k+1). \tag{5}$$

1.2 伪谱积分矩阵

令 $\{l_j(x)\}_{j=0}^N$ 为基于 Legendre-Guass-Lobatto 节点的 Lagrange 插值多项式, 即

$$\boldsymbol{D}^{(k)} = (d_{i,j}^{(k)})_{0 \leq i,j \leq N} = (l_j^k(x_i))_{0 \leq i,j \leq N}, \quad \boldsymbol{D}_{\text{in}}^{(k)} = (d_{i,j}^{(k)})_{1 \leq i,j \leq N-1}.$$

特别地, 记 $\boldsymbol{D} = \boldsymbol{D}^{(1)}$, $\boldsymbol{D}_{\text{in}} = \boldsymbol{D}_{\text{in}}^{(1)}$, 高阶微分矩阵可由一阶微分矩阵得到 (见文献[21]的定理 3.10), 即

$$\boldsymbol{D}^{(k)} = \boldsymbol{D} \boldsymbol{D} \cdots \boldsymbol{D} = \boldsymbol{D}^k.$$

记 $\boldsymbol{p}^k := (p^k(x_0), p^k(x_1), \cdots, p^k(x_{N-1}), p^k(x_N))^T$, $\boldsymbol{p} := \boldsymbol{p}^{(0)}$, 则可得伪谱分化过程, 即

$$\boldsymbol{D}^{(k)} \boldsymbol{p} = \boldsymbol{D}^k \boldsymbol{p} = \boldsymbol{p}^{(k)}, \quad k \geq 1.$$

考虑在内部 Gauss-Lobatto 节点的分化过程, 即

$$\hat{\boldsymbol{D}}^{(4)} \boldsymbol{p} = \hat{\boldsymbol{p}}^{(4)}. \tag{6}$$

式(6)中: $\hat{\boldsymbol{p}}^{(4)} = (p^{(4)}(-1), p^{(4)}(x_1), \cdots, p^{(4)}(x_{N-1}), p^{(4)}(1))^T$; $\hat{\boldsymbol{D}}^{(4)}$ 由 $\boldsymbol{e}_1 = (1, 0, \cdots, 0)$, $\boldsymbol{e}_N = (0, 0, \cdots, 1)$ 替换 $\boldsymbol{D}^{(4)}$ 的第一行和最后一行得到的.

基于适当的 Birkhoff 插值, 可得到 $\hat{\boldsymbol{D}}^{(4)}$ 的逆矩阵 \boldsymbol{B} , 由式(6)得 $\hat{\boldsymbol{B}} \hat{\boldsymbol{p}}^{(4)} = \boldsymbol{p}$, 即在内部 Gauss-Lobatto 点进行 4 次积分, 但端点处的函数值不变, 称 \boldsymbol{B} 为四阶伪谱积分矩阵.

1.3 Legendre-Gauss-Lobatto 点的 Birkhoff 插值

考虑式(2)的特殊性, 找 $p \in P_{N+2}^*$, 使得 $p^{(4)}(x_j) = u^{(4)}(x_j)$, $p(\pm 1) = u(\pm 1)$, $p'(\pm 1) = u'(\pm 1)$, $1 \leq j \leq N-1$. 则 u 的唯一 Birkhoff 插值多项式为

$$p(x) = u(-1)B_0(x) + \sum_{j=1}^{N-1} u^{(4)}(x_j)B_j(x) + u(1)B_N(x) + u'(-1)B_{N+1}(x) + u'(1)B_{N+2}(x). \tag{7}$$

式(7)中: $\{B_j\}_{j=0}^{N+2} \subset P_{N+2}^*$, 且满足如下条件

$$\left. \begin{aligned} B_0(-1) &= 1, & B_0(1) &= B'_0(\pm 1) = 0, & B_0^{(4)}(x_i) &= 0, & 1 \leq i \leq N-1, \\ B_N(1) &= 1, & B_N(-1) &= B'_N(\pm 1) = 0, & B_N^{(4)}(x_i) &= 0, & 1 \leq i \leq N-1, \\ B'_{N+1}(-1) &= 1, & B'_{N+1}(1) &= B_{N+1}(\pm 1) = 0, & B_{N+1}^{(4)}(x_i) &= 0, & 1 \leq i \leq N-1, \\ B'_{N+2}(1) &= 1, & B'_{N+2}(-1) &= B_{N+2}(\pm 1) = 0, & B_{N+2}^{(4)}(x_i) &= 0, & 1 \leq i \leq N-1, \\ B_j(\pm 1) &= B'_j(\pm 1) = 0, & B_j^{(4)}(x_i) &= \delta_{i,j}, & 1 \leq i, j \leq N-1. \end{aligned} \right\} \tag{8}$$

令 $b_{i,j}^{(k)} := B_j^{(k)}(x_i)$, 记 $\boldsymbol{B}^{(k)} = (b_{i,j}^{(k)})_{0 \leq i,j \leq N}$, $\boldsymbol{B}_{\text{in}}^{(k)} = (b_{i,j}^{(k)})_{1 \leq i,j \leq N-1}$.

特别地, $b_{i,j} := B_j(x_i)$, $\boldsymbol{B} = \boldsymbol{B}^{(0)}$, $\boldsymbol{B}_{\text{in}} = \boldsymbol{B}_{\text{in}}^{(0)}$, 则 $\boldsymbol{B}^{(k)} = \boldsymbol{D}^{(k)} \boldsymbol{B} = \boldsymbol{D}^k \boldsymbol{B} = \boldsymbol{D} \boldsymbol{B}^{(k-1)}$, $\hat{\boldsymbol{D}}^{(4)} \boldsymbol{B} = \boldsymbol{I}_{N+1}$, $\boldsymbol{D}_{\text{in}}^{(4)} \boldsymbol{B}_{\text{in}} = \boldsymbol{I}_{N-1}$. 为方便起见, 引入如下积分运算符^[20], 即

$$\partial_x^{-1}u(x)=\int_{-1}^xu(t)dt,\quad \partial_x^{-k}u(x)=\partial_x^{-1}(\partial_x^{-(k-1)}u(x)),\quad k\geqslant 2.$$

定理 1 满足条件(8)的 Birkhoff 插值基多项式,即

$$\left. \begin{aligned} B_0(x) &= \frac{1}{4}(x-1)^2(x+2), & B_N(x) &= \frac{1}{4}(x+1)^2(2-x), \\ B_{N+1}(x) &= \frac{1}{4}(x-1)^2(x+2), & B_{N+2}(x) &= \frac{1}{4}(x+1)^2(x-1), \\ B_j(x) &= \sum_{k=0}^{N-2}\beta_{k,j}\frac{\partial_x^{-4}L_x(x)}{\gamma_k}-\frac{1}{60}\beta_{3,j}(x^3-3x-2)-\frac{1}{12}\beta_{2,j}(x+1)^2+ \\ &\quad \frac{1}{20}\beta_{1,j}(2x^3+5x^5+4x+1)-\frac{1}{12}\beta_{0,j}x(x+1)^2, & j &= 1,2,\cdots,N-1. \end{aligned} \right\} \tag{9}$$

式(9)中: $\beta_{k,j}=(L_k(x_j)-\frac{1-(-1)^{N+k}}{2}L_{N-1}(x_j)-\frac{1+(-1)^{N+k}}{2}L_N(x_j))\omega_j$.

1.4 配点法

1.4.1 Birkhoff 配点法 用配点法求解边值问题(1),即找 $u_N\in P_{N+2}^*$,使得

$$\left. \begin{aligned} u_N^{(4)}(x_i)-\alpha u_N'(x_i)+\beta u_N(x_i) &= f(x_i), \\ u_N(\pm 1) &= u^\pm, \quad u_N'(\pm 1) = u_x^\pm, \quad 1\leqslant i\leqslant N-1. \end{aligned} \right\} \tag{10}$$

由式(7)可得 $u_N(x)$,Birkhoff 插值多项式展开为

$$u_N(x)=\sum_{j=1}^{N-1}u_N^{(4)}(x_j)B_j(x)+u^-B_0(x)+u^+B_N(x)+u_x^-B_{N+1}(x)+u_x^+B_{N+2}(x). \tag{11}$$

将式(11)代入式(10)中,线性系统为

$$(\boldsymbol{I}_{N-1}-\alpha\boldsymbol{B}_{\text{in}}^{(2)}+\beta\boldsymbol{B}_{\text{in}})\boldsymbol{V}=\tilde{\boldsymbol{f}}-\boldsymbol{u}_b. \tag{12}$$

式(12)中: $\boldsymbol{V}=(u_N^{(4)}(x_1),\cdots,u_N^{(4)}(x_{N-1}))^\text{T};\tilde{\boldsymbol{f}}=(f(x_1),f(x_2),\cdots,f(x_{N-1}))^\text{T};\boldsymbol{u}_b=\{\frac{u^-}{4}(\beta x_i^3-3(2\alpha+\beta)x_i+2\beta)+\frac{u_x^-}{4}(\beta x_i^3-\beta x_i^2-(6\alpha+\beta)x_i+(2\alpha+\beta))+\frac{u^+}{4}(-\beta x_i^3+3(2\alpha+\beta)x_i+2\beta)+\frac{u_x^+}{4}(\beta x_i^3+\beta x_i^2-(6\alpha+\beta)x_i-(2\alpha+\beta))\}_{i=1}^{N-1}$.

综上可得,Birkhoff 配点法求解式(1)的具体步骤为:

- 1) 根据定理求解出 \boldsymbol{B} 和 $\boldsymbol{B}^{(2)}$;
- 2) 由式(12)求解出 \boldsymbol{V} ;
- 3) 由式(11)求解出 $\boldsymbol{u}=(u_N(x_1),\cdots,u_N(x_{N-1}))^\text{T},\boldsymbol{u}=\boldsymbol{B}_{\text{in}}\boldsymbol{V}+u^-b_0+u^+b_N+u_x^-b_{N+1}+u_x^+b_{N+2}$. 其中: $b_j=(B_j(x_1),\cdots,B_j(x_{N-1}))^\text{T},j=0,N,N+1,N+2$.

1.4.2 Legendre 配点法 构造 u 的 Legendre 插值多项式,即

$$u_N(x)=\sum_{j=1}^{N-1}u_N(x_j)\varphi_j(x)+u^-\varphi_0(x)+u^+\varphi_N(x)+u_x^-\varphi_{N+1}(x)+u_x^+\varphi_{N+2}(x). \tag{13}$$

式(13)中: $\varphi_0(x)=\frac{1}{4}\{(x+2)(x-1)^2\};\varphi_N(x)=\frac{1}{4}\{(2-x)(x+1)^2\};\varphi_{N+1}(x)=\frac{1}{4}\{(x+1)(x-1)^2\};\varphi_{N+2}(x)=\frac{1}{4}\{(x-1)(x+1)^2\};\varphi_j(x)=d_j(\frac{2j+1}{2j+5}(L_{j+3}(x)-L_{j+1}(x))+L_{j-1}(x)-L_{j+1}(x)),1\leqslant j\leqslant N-1,d_j=\frac{1}{\sqrt{2(2j+1)^2(2j+3)}}$.

将式(13)代入式(10)中,线性系统为

$$(\boldsymbol{A}-\alpha\boldsymbol{C}+\beta\boldsymbol{D})\hat{\boldsymbol{V}}=\hat{\boldsymbol{f}}-\hat{\boldsymbol{u}}_b. \tag{14}$$

式(14)中: $\boldsymbol{A}=(\varphi_j^{(4)}(x_i))_{1\leqslant i,j\leqslant N-1};\boldsymbol{C}=(\varphi_j^{(2)}(x_i))_{1\leqslant i,j\leqslant N-1};\boldsymbol{D}=(\varphi_j(x_i))_{1\leqslant i,j\leqslant N-1};\hat{\boldsymbol{f}}=(f(x_1),f(x_2),\cdots,f(x_{N-1}))^\text{T};\hat{\boldsymbol{V}}=(u_N^{(4)}(x_1),\cdots,u_N^{(4)}(x_{N-1}))^\text{T};\hat{\boldsymbol{u}}_b=\{\frac{u^-}{2}(\beta-\beta x_i)+\frac{u_x^-}{4}(\beta x_i^3-\beta x_i^2-(6\alpha+\beta)x_i+(2\alpha+\beta))+\frac{u^+}{2}(\beta+\beta x_i)+\frac{u_x^+}{4}(\beta x_i^3+\beta x_i^2-(6\alpha+\beta)x_i-(2\alpha+\beta))\}_{i=1}^{N-1}$. 式(14)的求解过程与式(12)类似.

2 数值实验

例 1 当计算参数 $\alpha=\beta=1$ 时,取问题(1)在 $\Lambda=(-1,1)$ 上的解析解为 $u(x)=(1-x^2)^{9/2}$.

在 log-log 尺度下,分别由 Brikhoff 配点法(BCOL)和 Legendre 配点法(LCOL)得到的 L^2 -, H^1 -及 H^2 -误差(ϵ)随阶数(N)的变化,如图 1 所示. 由图 1 可知:Brikhoff 配点法和 Legendre 配点法得到的 L^2 -误差, H^1 -误差及 H^2 -误差随 N 的增大而衰减,且误差随 N 均呈代数收敛. 条件数随 N 的变化,如表 1 所示. 由表 1 可知:Birkhoff 配点法得到的代数方程组条件数较小,且随 N 的变化基本趋于一个稳定常数;而 Legendre 配点法对应的条件数较大,且随 N 的增大而快速增大,条件数的稳定性较差. 故 Birkhoff 配点法与 Legendre 配点法具有相同的精度,但由 Birkhoff 配点法得到的代数系统条件数比 Legendre 配点法得到的代数系统条件数更稳定.

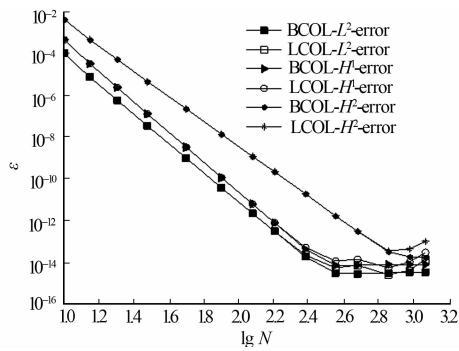


图 1 误差随 N 的变化情况
Fig. 1 Errors as a function of N

表 1 条件数随 N 的变化情况
Tab. 1 Condition number varies with N

N	16	32	64	128	256	512
BCOL 条件数	1.271 8	1.294 2	1.304 1	1.309 1	1.311 6	1.312 9
LCOL 条件数	1.214×10^3	2.374×10^4	4.997×10^5	1.090×10^7	2.422×10^8	5.431×10^9

例 2 当计算参数 $\alpha=\beta=1$ 时,取问题(1)在 $\Lambda=(-1,1)$ 上的解析解,即 $u(x)=\sin^2(k\pi x)$. 分别取 $k=1,2,4,6$ 进行计算,在半 log 尺度下,分别由 Brikhoff 配点法和 Legendre 配点法得到的 L^2 -误差, H^1 -误差及 H^2 -误差随 N 的变化情况,如图 2 所示. 由图 2 可知: L^2 -误差, H^1 -误差及 H^2 -误差均随 N 的增大而衰减,且误差随 N 呈指数收敛,但随着解析解的振荡性变大 Brikhoff 配点法在 N 取大时不受舍入误差的影响,而 Legendre 配点法则会受影响. 由于例 2 的线性系统的系数矩阵与例 1 相

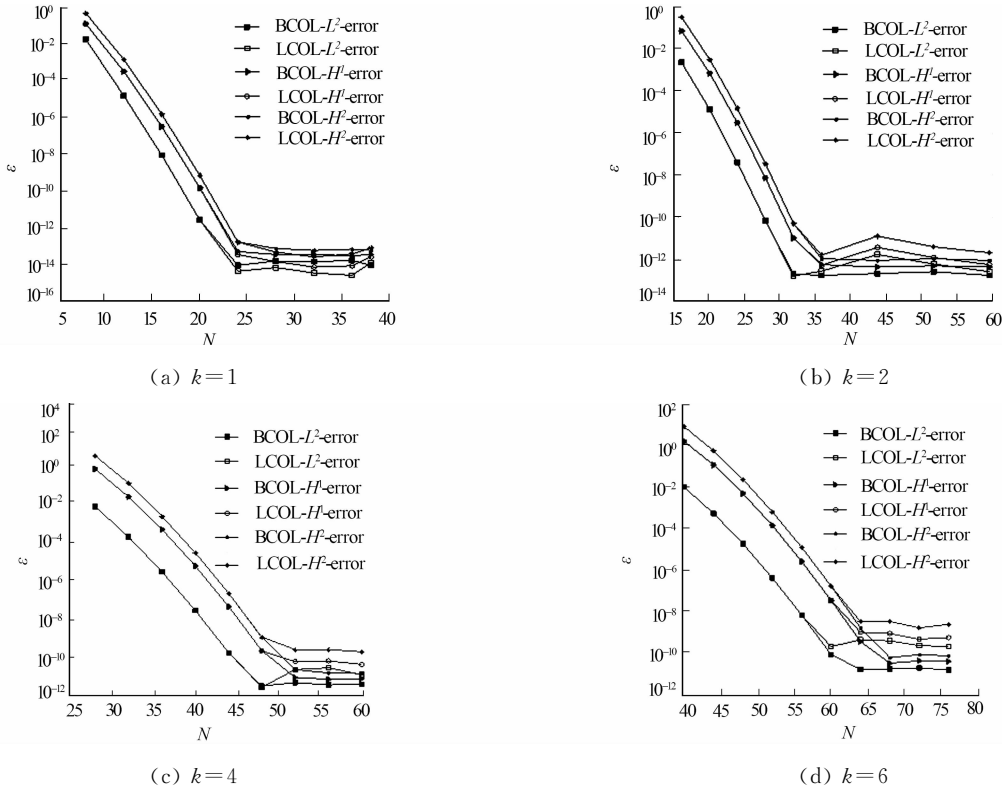


图 2 不同 k 值下的误差随 N 的变化情况
Fig. 2 Errors with different k values varies with N

同,由表 1 还可知:Birkhoff 配点法得到的条件数比 Legendre 配点法得到的条件数更稳定. 因此,Birkhoff 配点法对求解具有高频振荡解析解的四阶常微分方程比 Legendre 配点法更稳定.

例 3 考虑非线性 Kuramoto-Sivashinsky 问题,即

$$\begin{cases} \partial_t u(x,t) + u(x,t)\partial_x u(x,t) + \kappa \partial_x^2 u(x,t) + u(x,t) = 0, & (x,t) \in \Omega = I_x \times I_t, \\ u(x,0) = u_0(x), & x \in I_x, \\ u(a,t) = g_1(t), \quad u(b,t) = g_2(t), \quad u_x(a,t) = h_1(t), \quad u_x(b,t) = h_2(t), & t \in I_t. \end{cases}$$

对上述非线性方程采用如下格式进行离散,即

$$\frac{u_N^{k+1} - u_N^{k-1}}{2\tau} + \kappa \frac{\partial_x^2(u_N^{k+1} + u_N^{k-1})}{2} + \frac{\partial_x^4(u_N^{k+1} + u_N^{k-1})}{2} = -u_N^k \partial_x u_N^k.$$

可得到 $\partial_x^4 u_N^{k+1} + \kappa \partial_x^2 u_N^{k+1} + \frac{u_N^{k+1}}{\tau} = F$. 其中: $F = -2u_N^k \partial_x u_N^k - \partial_x^2 u_N^{k-1} - \kappa \partial_x^2 u_N^{k-1} + (u_N^{k-1})/\tau, k=1,2,\cdots, (t/\tau), \tau$ 为时间步长. 因此,每个时间步只需求解类似式(12)的线性系统即可. $k=0$ 时,采用如下格式求解 u_N^1 ,即

$$u_N^1 = u_0 + \tau(-u_0 \partial_x u_0 + \partial_x^2 u_0 - \partial_x^4 u_0).$$

当 $\kappa=-1, I_x=(-50,50), I_t=(0,t)$ 时,取解析解为

$$u(x,t) = c + \frac{15}{19\sqrt{19}}(-3\tanh(k(x-ct-x_0)) + \tanh^3(k(x-ct-x_0))).$$

上式中:取 $c=5, k=1/(2\sqrt{19}), x_0=-25$ 进行计算. 当 $N=100$ 时,不同时间步长计算得到的 H^2 -误差,如表 2 所示. 由表 2 可知:数值解在时间方向具有二阶收敛率.

表 2 $N=100$ 时的误差					
Tab. 2 Errors with $N=100$					
τ	0.1	0.01	0.001	0.000 1	0.000 01
H^2 -误差	1.532×10^{-3}	1.604×10^{-5}	1.614×10^{-7}	1.621×10^{-9}	4.832×10^{-10}

当 $\kappa=1, I_x=(-30,30), I_t=(0,4)$ 时,解析解为

$$u(x,t) = c + \frac{15}{19\sqrt{11/19}}(-9\tanh(k(x-ct-x_0)) + 11\tanh^3(k(x-ct-x_0))).$$

上式中: $c=5, k=1/(2\sqrt{11/19}), x_0=-12$. 当 $N=100, \tau=0.004$ 时,方程的精确解与数值解的时空对比,如图 3 所示. 由图 3 可知:数值解拟合效果良好,说明 Birkhoff 配点法求解非线性方程也是有效的.

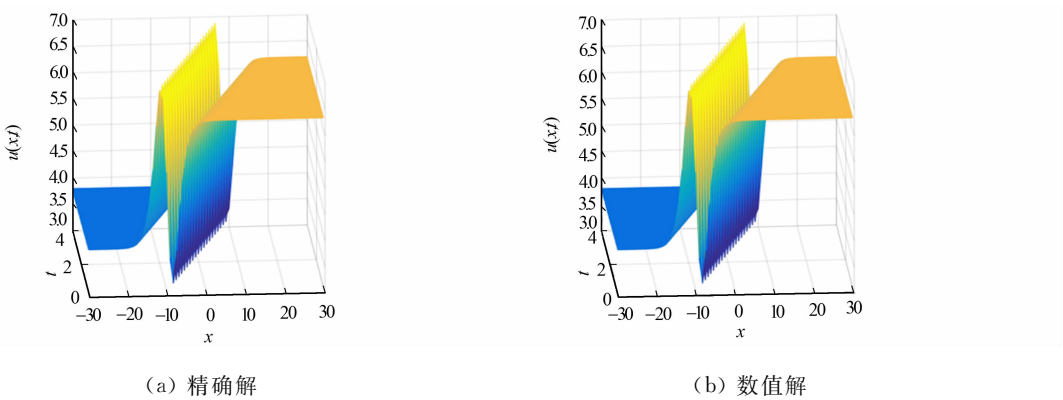


图 3 $I_t=(0,4)$ 时精确解与数值解的时空图

Fig. 3 Spatiotemporal maps of numerical solutions and exact solutions with $I_t=(0,4)$

3 结束语

Birkhoff 配点法求解四阶微分方程. 根据插值条件得到 Birkhoff 插值多项式,建立具有适定条件的 Birkhoff 配点法,得到由伪谱积分矩阵表示的具有稳定条件数的线性系统,并给出 Legendre 配点法得到的线性系统. 通过算例说明:Birkhoff 配点法所得到的线性系统具有稳定的条件数,且具有高精度,同

时, Birkhoff 配点法适用于求解一些常系数、具有振荡解析解、时间依赖性及非线性的四阶微分方程。

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