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经典的 Drinfel'd-Sokolov-Wilson 方程 的非线性波解

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摘要: 利用 (G'/G) -展开法, 构造经典的 Drinfel'd-Sokolov-Wilson 方程的新的非线性波解. 这些非线性波解分别以双曲函数、三角函数和分式函数的形式表达. 结果表明: (G'/G) -展开法是研究数学物理方程的非线性波解的一种有效工具.

关键词: Drinfel'd-Sokolov-Wilson 方程; (G'/G) -展开法; 非线性波解; 显式表达式

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Nonlinear Wave Solutions for the Classical Drinfel'd-Sokolov-Wilson Equation

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Abstract: We constructed new nonlinear wave solutions for the classical Drinfel'd-Sokolov-Wilson Equation by exploiting (G'/G) -expansion method. These nonlinear wave solutions are expressed in the forms of the hyperbolic functions, the trigonometric functions and the rational functions. The results show that (G'/G) -expansion method is an efficient tool for studying nonlinear wave solutions of mathematical physics equations.

Keywords: Drinfel'd-Sokolov-Wilson equation; (G'/G) -expansion method; nonlinear wave solutions; explicit expressions

数学物理方程的解有助于加深对其所描述的自然现象或过程的理解和认识. 因此, 寻找数学物理方程的非线性波解是数学物理工作者研究的热点问题.

经典的 Drinfel'd-Sokolov-Wilson(DSW)方程

$$\left. \begin{aligned} u_t + pvv_x &= 0, \\ v_t + ruv_x + su_xv + qv_{xxx} &= 0 \end{aligned} \right\} \quad (1)$$

被引入后, 它及其变体得到了人们的广泛关注^[1-7]. 式(1)中: p, q, r, s 都是非零常数.

Hirota 等^[1]给出了 DSW 方程(1)的一种特殊形式

$$\left. \begin{aligned} u_t + 3uv_x &= 0, \\ v_t + 2uv_x + 2u_xv + 2v_{xxx} &= 0 \end{aligned} \right\} \quad (2)$$

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的孤子结构. 文献[4-6]分别用代数方法、改进的广义 Jacobi 椭圆函数方法和改进的 F-展开法找到了方程(2)的一些精确行波解. 对于 DSW 方程(1), 文献[2-3]分别利用直接的代数方法获得了一些精确行波解. 此外, Wen 等^[7]利用微分方程定性理论和动力系统分支方法^[7-17]给出了 DSW 方程(1)的 30 个精确行波解. 前面的工作已找到了一部分解, 但新的解仍有待发现. 本文利用 (G'/G) -展开法^[18-19]研究 DSW 方程(1)的非线性波解, 得到了一些新的解.

1 用 (G'/G) -展开法求解 DSW 方程(1)

对方程(1)进行替换, $u(x, t) = u(\xi), v(x, t) = v(\xi), \xi = x - ct$, 可得

$$\left. \begin{aligned} -cu' + pvv' &= 0, \\ -cv' + ruv' + su'v + qv''' &= 0. \end{aligned} \right\} \tag{3}$$

假定 $u(\xi)$ 和 $v(\xi)$ 可以展开成关于 (G'/G) 的多项式, 即

$$u(\xi) = \sum_{i=0}^m a_i (G'/G)^i, \quad a_m \neq 0, \tag{4}$$

$$v(\xi) = \sum_{j=0}^n b_j (G'/G)^j, \quad b_n \neq 0. \tag{5}$$

式(4), (5)中: a_0, a_1, \dots, a_m 和 b_0, b_1, \dots, b_n 是待确定的常数, 且 $G = G(\xi)$ 满足二阶常微分方程

$$G'' + \lambda G' + \mu G = 0. \tag{6}$$

式(6)中: λ 和 μ 是常数.

利用 u' 与 vv' , 以及 $u'v, uv'$ 与 v''' 之间的齐次平衡, 得到 $m = 2, n = 1$. 式(4), (5)可以表示为

$$u(\xi) = a_0 + a_1 (G'/G) + a_2 (G'/G)^2, \quad a_2 \neq 0, \tag{7}$$

$$v(\xi) = b_0 + b_1 (G'/G), \quad b_1 \neq 0. \tag{8}$$

把式(7), (8)代入方程(3)中, 可得

$$\begin{aligned} &\mu(ca_1 - pb_0b_1) + (\lambda(ca_1 - pb_0b_1) + \mu(2ca_2 - pb_1^2))(G'/G) + \\ &(ca_1 - pb_0b_1 + \lambda(2ca_2 - pb_1^2))(G'/G)^2 + (2ca_2 - pb_1^2)(G'/G)^3 = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &\mu((sa_1b_0 + ra_0b_1) + b_1(q\lambda^2 + 2q\mu - c)) + (\lambda((sa_1b_0 + ra_0b_1) + b_1(q\lambda^2 + 2q\mu - c)) + \\ &(2sa_2b_0 + 6q\lambda b_1 + a_1b_1(r + s)))(G'/G) + ((sa_1b_0 + ra_0b_1) + \\ &b_1(q\lambda^2 + 2q\mu - c) + \lambda(2sa_2b_0 + 6q\lambda b_1 + a_1b_1(r + s)) + \\ &\mu b_1(6q + a_2(r + 2s)))(G'/G)^2 + ((2sa_2b_0 + 6q\lambda b_1 + (r + s)a_1b_1) + \\ &\lambda b_1(6q + a_2(r + 2s)))(G'/G)^3 + b_1(6q + a_2(r + 2s))(G'/G)^4 = 0. \end{aligned} \tag{10}$$

令式(9), (10)中, $(G'/G)^k (k = 0, 1, 2, 3, 4)$ 的系数为零, 得到

$$\left. \begin{aligned} ca_1 - pb_0b_1 &= 0, \\ 2ca_2 - pb_1^2 &= 0, \\ (sa_1b_0 + ra_0b_1) + b_1(q\lambda^2 + 2q\mu - c) &= 0, \\ 2sa_2b_0 + 6q\lambda b_1 + a_1b_1(r + s) &= 0, \\ 6q + a_2(r + 2s) &= 0. \end{aligned} \right\} \tag{11}$$

为方便起见, 令 $\theta = c(r + 2s) - q(\lambda^2(r + s) - 2\mu(r + 2s))$, 则方程组(11)的解为

$$\left. \begin{aligned} a_0 &= \frac{\theta}{r(r + 2s)}, \quad a_1 = -\frac{6q\lambda}{r + 2s}, \quad a_2 = -\frac{6q}{r + 2s}, \\ b_0 &= \lambda\sqrt{\frac{-3qc}{p(r + 2s)}}, \quad b_1 = \sqrt{\frac{-12qc}{p(r + 2s)}}, \end{aligned} \right\} \tag{12}$$

或

$$\left. \begin{aligned} a_0 &= \frac{\theta}{r(r + 2s)}, \quad a_1 = -\frac{6q\lambda}{r + 2s}, \quad a_2 = -\frac{6q}{r + 2s}, \\ b_0 &= -\lambda\sqrt{\frac{-3qc}{p(r + 2s)}}, \quad b_1 = -\sqrt{\frac{-12qc}{p(r + 2s)}}. \end{aligned} \right\} \tag{13}$$

把式(12), (13)分别代入式(7), (8), 得到方程(1)的一般形式的解, 即

$$\left. \begin{aligned} u(\xi) &= \frac{1}{r+2s} \left(\frac{\theta}{r} - 6q\lambda \left(\frac{G'}{G} \right) - 6q \left(\frac{G'}{G} \right)^2 \right), \\ v(\xi) &= \pm \sqrt{\frac{-3qc}{p(r+2s)}} \left(\lambda + 2 \left(\frac{G'}{G} \right) \right). \end{aligned} \right\} \quad (14)$$

2 DSW 方程(1)的非线性波解

定理 1 方程(1)有如下形式的显式非线性波解.

1) 当 $\lambda^2 - 4\mu = 0$ 时, 有

$$\left. \begin{aligned} u(x, t) &= \frac{1}{r+2s} \left(\frac{\theta}{r} + \frac{3q\lambda^2}{2} \right) - \frac{6q}{(r+2s)} \left(\frac{c_2}{c_1 + c_2(x-ct)} \right)^2, \\ v(x, t) &= \pm \sqrt{\frac{-3qc(\lambda^2 - 4\mu)}{p(r+2s)}} \frac{2c_2}{c_1 + c_2(x-ct)}. \end{aligned} \right\} \quad (15)$$

2) 当 $\lambda^2 - 4\mu > 0$ 时, 有

$$\left. \begin{aligned} u(x, t) &= \frac{1}{r+2s} \left(\frac{\theta}{r} + \frac{3q\lambda^2}{2} \right) - \frac{3q}{2(r+2s)} (\lambda^2 - 4\mu) \times \\ &\quad \left\{ \frac{c_1 \sinh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right) + c_2 \cosh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right)}{c_1 \cosh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right) + c_2 \sinh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right)} \right\}^2, \\ v(x, t) &= \pm \sqrt{\frac{-3qc(\lambda^2 - 4\mu)}{p(r+2s)}} \times \\ &\quad \frac{c_1 \sinh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right) + c_2 \cosh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right)}{c_1 \cosh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right) + c_2 \sinh\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right)}. \end{aligned} \right\} \quad (16)$$

3) 当 $\lambda^2 - 4\mu < 0$ 时, 有

$$\left. \begin{aligned} u(x, t) &= \frac{1}{r+2s} \left(\frac{\theta}{r} + \frac{3q\lambda^2}{2} \right) - \frac{3q}{2(r+2s)} (4\mu - \lambda^2) \times \\ &\quad \left\{ \frac{-c_1 \sin\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right) + c_2 \cos\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right)}{c_1 \cos\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right) + c_2 \sin\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right)} \right\}^2, \\ v(x, t) &= \pm \sqrt{\frac{-3qc(4\mu - \lambda^2)}{p(r+2s)}} \times \\ &\quad \frac{-c_1 \sin\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right) + c_2 \cos\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right)}{c_1 \cos\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right) + c_2 \sin\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right)}. \end{aligned} \right\} \quad (17)$$

式(15)~(17)中: c_1 和 c_2 是任意常数.

证明 根据常微分方程理论, 易得方程(6)的解.

1) 当 $\lambda^2 - 4\mu = 0$ 时, 有

$$G = \exp\left(-\frac{\lambda}{2}(x-ct)\right)(c_1 + c_2(x-ct)). \quad (18)$$

2) 当 $\lambda^2 - 4\mu > 0$ 时, 有

$$G = \exp\left(-\frac{\lambda}{2}(x-ct)\right) \left(c_1 \exp\left(\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right) + c_2 \exp\left(-\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x-ct)\right) \right). \quad (19)$$

3) 当 $\lambda^2 - 4\mu < 0$ 时, 有

$$G = \exp\left(-\frac{\lambda}{2}(x-ct)\right) \left(c_1 \cos\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right) + c_2 \sin\left(\frac{1}{2} \sqrt{4\mu - \lambda^2}(x-ct)\right) \right). \quad (20)$$

将式(18)~(20)分别代入方程(14),即得到非线性波解(15)~(17). 证毕.

3 结 束 语

利用 (G'/G) -展开法,构造了经典的 Drinfel'd-Sokolov-Wilson 方程的新的非线性波解. 这些非线性波解具有丰富的结构,分别以双曲函数、三角函数和分式函数的形式给出. 此外,当参数取一些特殊的值时,这些非线性波解展现出不同类型的波形,包括孤立波、奇异波、周期奇异波等.

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