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广义 Lorenz 系统的 Painlevé 分析及其精确解

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摘要: 考虑一个 Hamilton 函数为 $H=\frac{1}{2}\sigma y^2-\sigma xy+rx y u+\frac{x^2}{2}z-\frac{\rho}{2}x^2-\beta uz$ 的四维广义 Lorenz 系统, 利用 Painlevé 分析的方法, 将该系统进行奇异流型展开. 利用调谐因子项将其进行有限项“截断”, 证明其具有 Painlevé 可积性, 并导出其自 Bäcklund 变换和奇异流型满足的 Schwarz 导数方程. 通过研究相关的 Schwarz 导数方程的性质, 求出广义 Lorenz 系统的精确解.

关键词: 广义 Lorenz 系统; Painlevé 分析; 调谐因子; Bäcklund 变换; Schwarz 导数

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1 预备知识

混沌是非线性系统研究领域非常活跃的前沿课题^[1]. Painlevé 方法已被广泛应用于探讨非线性微分方程的可积性, 并被证明是一种十分有效的途径^[2-10]. 将 Lorenz 系统推广为四维广义 Lorenz 系统, 即

$$\left. \begin{aligned} \dot{x} &= \sigma(y-x) + rxu, \\ \dot{y} &= \rho x + \sigma y - xz - r y u, \\ \dot{z} &= \beta z - r x y, \\ \dot{u} &= x^2/2 - \beta u. \end{aligned} \right\} \tag{1}$$

系统(1)中: $\sigma, \rho, \beta, \gamma$ 是常量; x, y, z, u 是自变量 t 的函数; $\dot{x}, \dot{y}, \dot{z}, \dot{u}$ 分别表示 x, y, z, u 关于 t 的导函数. 这是一个 2 维 Hamilton 系统, 即

$$H = \frac{1}{2}\sigma y^2 - \sigma xy + rx y u + \frac{x^2}{2}z - \frac{\rho}{2}x^2 - \beta uz.$$

对 x, y, z, u 进行奇异流型展开, 可得

$$\left. \begin{aligned} x &= \sum_{k=0}^{\infty} a_k \phi^{k-a_1}, & y &= \sum_{k=0}^{\infty} b_k \phi^{k-a_2}, \\ z &= \sum_{k=0}^{\infty} c_k \phi^{k-a_3}, & u &= \sum_{k=0}^{\infty} d_k \phi^{k-a_4}. \end{aligned} \right\} \tag{2}$$

式(2)中: ϕ 为奇异流型. 通过主导项分析得 $\alpha_1=\alpha_4=1, \alpha_2=\alpha_3$; 且有

$$\left. \begin{aligned} a_0 &= \sqrt{\frac{2}{r}}\phi', & b_0 &= e_1\phi', \\ c_0 &= \sqrt{2r}e_1\phi', & d_0 &= -\frac{\phi'}{r}, & e_1 &= \text{const.} \end{aligned} \right\} \tag{3}$$

令 $\alpha_2=\alpha_3=1$, 即

$$x = \sum_{k=0}^{\infty} a_k \phi^{k-1}, \quad y = \sum_{k=0}^{\infty} b_k \phi^{k-1}, \quad z = \sum_{k=0}^{\infty} c_k \phi^{k-1}, \quad u = \sum_{k=0}^{\infty} d_k \phi^{k-1}.$$

将其代入系统(1), 有

$$\sum_{k=0}^{\infty} (a'_k \phi^{k-1} + (k-1)a_k \phi^{k-2} \phi') = \sigma \sum_{k=0}^{\infty} (b_k - a_k) \phi^{k-1} + r \sum_{k=0}^{\infty} a_k \phi^{k-1} \sum_{j=0}^{\infty} d_j \phi^{j-1}, \quad (4)$$

$$\begin{aligned} \sum_{k=0}^{\infty} (b'_k \phi^{k-1} + (k-1)b_k \phi^{k-2} \phi') &= \rho \sum_{k=0}^{\infty} a_k \phi^{k-1} + \sigma \sum_{k=0}^{\infty} b_k \phi^{k-1} - \\ &\sum_{k=0}^{\infty} a_k \phi^{k-1} \sum_{j=0}^{\infty} c_j \phi^{j-1} - r \sum_{k=0}^{\infty} b_k \phi^{k-1} \sum_{j=0}^{\infty} d_j \phi^{j-1}, \end{aligned} \quad (5)$$

$$\sum_{k=0}^{\infty} (c'_k \phi^{k-1} + (k-1)c_k \phi^{k-2} \phi') = \beta \sum_{k=0}^{\infty} c_k \phi^{k-1} - r \sum_{k=0}^{\infty} a_k \phi^{k-1} \sum_{j=0}^{\infty} b_j \phi^{j-1}, \quad (6)$$

$$\sum_{k=0}^{\infty} (d'_k \phi^{k-1} + (k-1)d_k \phi^{k-2} \phi') = \left(\sum_{k=0}^{\infty} a_k \phi^{k-1} \right)^2 / 2 - \beta \sum_{k=0}^{\infty} d_k \phi^{k-1}. \quad (7)$$

比较 ϕ 的同次幂系数, 得

$$\left. \begin{aligned} \phi^{-2}: a_0 \phi' &= -ra_0 b_0, & b_0 \phi' &= a_0 c_0 + rb_0 d_0, \\ c_0 \phi' &= ra_0 b_0, & d_0 \phi' &= -\frac{1}{2}a_0^2; \end{aligned} \right\} \quad (8)$$

$$\left. \begin{aligned} \phi^{-1}: a'_0 &= \sigma(b_0 - a_0) + r(a_0 d_1 + a_1 d_0), \\ b'_0 &= \rho a_0 + \sigma b_0 - (a_0 c_1 + a_1 c_0) - r(b_0 d_1 + b_1 d_0), \\ c'_0 &= \beta c_0 - r(a_0 b_1 + a_1 b_0), & d'_0 &= a_0 a_1 - \beta d_0; \end{aligned} \right\} \quad (9)$$

$$\left. \begin{aligned} \phi^0: a'_1 + a_2 \phi' &= \sigma(b_1 - a_1) + r(a_1 d_1 + a_0 d_2 + a_2 d_0), \\ b'_1 + b_2 \phi' &= \rho a_1 + \sigma b_1 - (a_1 c_1 + a_0 c_2 + a_2 c_0) - r(b_1 d_1 + b_0 d_2 + b_2 d_0), \\ c'_1 + c_2 \phi' &= \beta c_1 - r(a_1 b_1 + a_0 b_2 + a_2 b_0), & d'_1 + d_2 \phi' &= \frac{1}{2}a_1^2 - \beta d_1; \end{aligned} \right\} \quad (10)$$

$$\left. \begin{aligned} \phi^{k-2}: a'_{k-1} + (k-1)a_k \phi' &= \sigma(b_{k-1} - a_{k-1}) + r \sum_{j=0}^k a_j d_{k-j}, \\ b'_{k-1} + (k-1)b_k \phi' &= \rho a_{k-1} + \sigma b_{k-1} - \sum_{j=0}^k a_j c_{k-j} - r \sum_{j=0}^k b_j d_{k-j}, \\ c'_{k-1} + (k-1)c_k \phi' &= \beta c_{k-1} - r \sum_{j=0}^k a_j b_{k-j}, \\ d'_{k-1} + (k-1)d_k \phi' &= \sum_{j=0}^k a_j a_{k-j} - \beta d_{k-1}. \end{aligned} \right\} \quad (11)$$

计算调谐因子, 对式(11)移项可得

$$\left. \begin{aligned} a_k [(k-1)\phi' - rd_0] + d_k (-ra_0) &= \sigma(b_{k-1} - a_{k-1}) + r \sum_{j=1}^{k-1} a_j b_{k-j} - a'_{k-1}, \\ c_0 a_k + b_k [(k-1)\phi' + rd_0] + a_0 c_k + ra_0 d_k &= \rho a_{k-1} + \sigma b_{k-1} - \sum_{j=1}^{k-1} a_j c_{k-j} - r \sum_{j=1}^{k-1} b_j d_{k-j} - b'_{k-1}, \\ rb_0 a_k + ra_0 b_k + (k-1)c_k \phi' &= \beta c_{k-1} - r \sum_{j=1}^{k-1} a_j b_{k-j} - c'_{k-1}, \\ -2a_0 a_k + (k-1)d_k \phi' &= \sum_{j=1}^{k-1} a_j a_{k-j} - \beta d_{k-1} - d'_{k-1}. \end{aligned} \right\} \quad (12)$$

分别记式(12)中各个方程的右式为 F_1, F_2, F_3, F_4 , 其中 F_1, F_2, F_3, F_4 为 $a_{k-1}, b_{k-1}, c_{k-1}, d_{k-1}, \dots, a_0, b_0, c_0, d_0, \phi$ 的函数. 将式(3)代入式(12), 转换成矩阵形式得

$$\phi' = \begin{bmatrix} k & 0 & 0 & -\sqrt{2}r \\ \sqrt{2}re_1 & k-2 & \sqrt{2}/r & re_1 \\ re_1 & \sqrt{2}r & k-1 & 0 \\ -2\sqrt{2}/r & 0 & 0 & k-1 \end{bmatrix} \begin{bmatrix} a_k \\ b_k \\ c_k \\ d_k \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}. \quad (13)$$

利用行列式计算调谐因子,可得

$$\begin{vmatrix} k & 0 & 0 & -\sqrt{2r} \\ \sqrt{2r}e_1 & k-2 & \sqrt{2/r} & re_1 \\ re_1 & \sqrt{2r} & k-1 & 0 \\ -2\sqrt{2/r} & 0 & 0 & k-1 \end{vmatrix} = k(k-3)(k^2-k-4), \tag{14}$$

得到调谐因子 $k=0,3$. 若令 $a_2=b_2=c_2=d_2=0$, 经过计算可推出当 $j\geqslant 3$ 时, $a_j=b_j=c_j=d_j=0$, 即 Painlevé 奇异流型展式(2)被成功“截断”, 从而该四维广义 Lorenz 系统(1)具有 Painlevé 可积性.

命题 1 四维广义的 Lorenz 系统(1)有 Bäcklund 变换, 即

$$\left. \begin{aligned} x &= a_0\phi^{-1} + a_1, & y &= b_0\phi^{-1} + b_1, \\ z &= c_0\phi^{-1} + c_1, & u &= d_0\phi^{-1} + d_1. \end{aligned} \right\} \tag{15}$$

式(15)中, ϕ 满足 Schwarz 方程

$$\{\phi;t\} = \frac{3}{2}\beta^2 + \beta\sigma c_1\sqrt{2r} - 2\sigma\beta = \text{const.} \tag{16}$$

证明 利用式(8)~(10), 经计算可得

$$\left. \begin{aligned} a_0 &= \sqrt{2/r}\phi', & b_0 &= e_1\phi', \\ c_0 &= \sqrt{2r}e_1\phi', & d_0 &= -\phi'/r, & e_1 &= \text{const.} \end{aligned} \right\} \tag{17}$$

$$\left. \begin{aligned} a'_1 &= \sigma(b_1 - a_1) + ra_1d_1, & b'_1 &= \rho a_1 + \sigma b_1 - a_1c_1 - rb_1d_1, \\ c'_1 &= \beta c_1 - ra_1b_1, & d'_1 &= a_1^2/2 - \beta d_1. \end{aligned} \right\} \tag{18}$$

$$\left. \begin{aligned} a_1 &= -\frac{1}{\sqrt{2r}}\frac{\phi''}{\phi'} - \frac{\beta}{\sqrt{2r}}, & b_1 &= \frac{3}{2}e_1\beta - \frac{e_1}{2}\frac{\phi''}{\phi'}, \\ c_1 &= \rho + \frac{r}{2}e_1^2\sigma + 3\beta e_1\sqrt{\frac{r}{2}} - e_1\sqrt{\frac{r}{2}}\frac{\phi''}{\phi'}, \\ d_1 &= \frac{1}{2r}\frac{\phi''}{\phi'} - \frac{\sigma e_1}{\sqrt{2r}} + \frac{\sigma}{r} - \frac{\beta}{2r}. \end{aligned} \right\} \tag{19}$$

$$\{\phi;t\} = \frac{3}{2}\beta^2 + \beta\sigma c_1\sqrt{2r} - 2\sigma\beta. \tag{20}$$

其中: $\{\phi;t\} = \frac{\phi'''}{\phi} - \frac{3}{2}\frac{\phi''^2}{\phi'^2}$ 是 Schwarz 方程.

由式(18)可知: a_1, b_1, c_1, d_1 满足系统(1), 同时 x, y, z, u 满足系统(1), 故式(15)为系统(1)的自 Bäcklund 变换.

2 广义 Lorenz 系统的精确解

通过求解 Schwarz 导数方程, 得到该系统的一个精确解. 设 $\frac{3}{2}\beta^2 + \beta\sigma c_1\sqrt{2r} - 2\sigma\beta = \lambda$, 则

$$\frac{\phi'''}{\phi} - \frac{3}{2}\frac{\phi''^2}{\phi'^2} = \lambda, \tag{21}$$

将其变形为 $(\frac{\phi''}{\phi})' = \lambda + \frac{1}{2}(\frac{\phi''}{\phi})^2$. 设 $y = \frac{\phi''}{\phi}$, 可得

$$y' = \lambda + \frac{y^2}{2}. \tag{22}$$

1) 当 $\lambda\geqslant 0$ 时, 从式(22)可求解得

$$y = \sqrt{2\lambda}i \cdot \frac{1 + \exp(\sqrt{2\lambda}i(t+c))}{1 - \exp(\sqrt{2\lambda}i(t+c))}.$$

因为 $\frac{\phi''}{\phi} = (\ln \phi')'$, 所以 $\ln \phi' = \sqrt{2\lambda}it + c_2 - 2\ln(1 - \exp(\sqrt{2\lambda}i(t+c_1)))$. 经变换可得

$$\phi' = \frac{\exp(\sqrt{2\lambda}i(t+c_2))}{(1-\exp(\sqrt{2\lambda}i(t+c_1)))^2}.$$

两边积分后, 可得

$$\phi = \frac{c_2}{\sqrt{2\lambda}i} \cdot \frac{1}{1-\exp(\sqrt{2\lambda}i(t+c_1))}. \quad (23)$$

2) 当 $\lambda < 0$ 时, 从式(22)可求解为

$$y = \sqrt{-2\lambda} \cdot \frac{1+\exp(\sqrt{-2\lambda}(t+c))}{1-\exp(\sqrt{-2\lambda}(t+c))},$$

因为 $\frac{\phi''}{\phi} = (\ln \phi')'$, 所以 $\ln \phi' = \sqrt{-2\lambda}t + c_2 - 2\ln(1-\exp(\sqrt{-2\lambda}(t+c_1)))$. 经变换可得

$$\phi' = \frac{\exp(\sqrt{-2\lambda}(t+c_2))}{(1-\exp(\sqrt{-2\lambda}(t+c_1)))^2}.$$

两边积分后可得

$$\phi = \frac{c_2}{\sqrt{-2\lambda}} \cdot \frac{1}{1-\exp(\sqrt{-2\lambda}(t+c_1))}. \quad (24)$$

综上所述, 当 $\lambda \geq 0$ 时, $\phi = \frac{c_2}{\sqrt{2\lambda}i} \cdot \frac{1}{1-\exp(\sqrt{2\lambda}i(t+c_1))}$; 而当 $\lambda < 0$ 时, $\phi = \frac{c_2}{\sqrt{-2\lambda}} \cdot \frac{1}{1-\exp(\sqrt{-2\lambda}(t+c_1))}$. 其中 ϕ 为指数形式的解. 从而可得该广义 Lorenz 系统的一个精确解为

$$\left. \begin{aligned} x &= \sqrt{\frac{2}{r}}\phi'\phi^{-1} - \frac{1}{\sqrt{2r}}\frac{\phi''}{\phi'} - \frac{\beta}{\sqrt{2r}}, \\ y &= e_1\phi'\phi^{-1} + \frac{3}{2}e_1\beta - \frac{e_1}{2}\frac{\phi''}{\phi'}, \\ z &= \sqrt{2r}e_1\phi'\phi^{-1} + \rho + \frac{r}{2}e_1^2\sigma + 3\beta e_1\sqrt{\frac{r}{2}} - e_1\sqrt{\frac{r}{2}}\frac{\phi''}{\phi'}, \\ u &= -\frac{\phi'}{r}\phi^{-1} + \frac{1}{2r}\frac{\phi''}{\phi'} - \frac{\sigma e_1}{\sqrt{2r}} + \frac{\sigma}{r} - \frac{\beta}{2r}. \end{aligned} \right\} \quad (25)$$

其中 ϕ 已由上面解出.

引理 1 设 y_1, y_2 为常微分方程 $\frac{d^2y}{dt^2} + Q(t)y = 0$ 的两个线性无关解, 则 $w = \frac{y_1}{y_2}$ 满足 Schwarz 导数方程 $\{w; t\} = 2Q(t)$.

令 $Q(t) = \lambda/2$, 其中 $\lambda = \frac{3}{2}\beta^2 + \beta\sigma c_1\sqrt{2r} - 2\sigma\beta$. 由引理 1 可得 $\frac{d^2y}{dt^2} + Q(t)y = 0$ 的两个线性无关解.

1) 当 $\lambda \geq 0$ 时, 有

$$y_1 = C_1 \exp(\sqrt{\frac{\lambda}{2}}it) + C_2 \exp(-\sqrt{\frac{\lambda}{2}}it),$$

$$y_2 = C_3 \exp(\sqrt{\frac{\lambda}{2}}it) + C_4 \exp(-\sqrt{\frac{\lambda}{2}}it).$$

2) 当 $\lambda < 0$ 时, 有

$$y_1 = C_1 \exp(\sqrt{-\frac{\lambda}{2}}t) + C_2 \exp(-\sqrt{-\frac{\lambda}{2}}t),$$

$$y_2 = C_3 \exp(\sqrt{-\frac{\lambda}{2}}t) + C_4 \exp(-\sqrt{-\frac{\lambda}{2}}t).$$

其中 C_1, C_2, C_3, C_4 为任意常数, 而且满足关系式 $C_1C_4 \neq C_2C_3$. 从而, 有 $w = y_1/y_2$, 满足 $\{w; t\} = \lambda$.

综上所述, 当 $\lambda \geq 0$ 时, $\phi = \frac{C_1 \exp(\sqrt{\frac{\lambda}{2}}it) + C_2 \exp(-\sqrt{\frac{\lambda}{2}}it)}{C_3 \exp(\sqrt{\frac{\lambda}{2}}it) + C_4 \exp(-\sqrt{\frac{\lambda}{2}}it)}$; 而当 $\lambda < 0$ 时, $\phi =$

$$\frac{C_1 \exp(\sqrt{-\frac{\lambda}{2}} t) + C_2 \exp(-\sqrt{\frac{\lambda}{2}} t)}{C_3 \exp(\sqrt{-\frac{\lambda}{2}} t) + C_4 \exp(-\sqrt{\frac{\lambda}{2}} t)}.$$

从而可得该广义 Lorenz 系统的另一个精确解为

$$\left. \begin{aligned} x &= \sqrt{\frac{2}{r}} \phi' \phi^{-1} + -\frac{1}{\sqrt{2r}} \frac{\phi''}{\phi} - \frac{\beta}{\sqrt{2r}}, \\ y &= e_1 \phi' \phi^{-1} + \frac{3}{2} e_1 \beta - \frac{e_1}{2} \frac{\phi''}{\phi}, \\ z &= \sqrt{2r} e_1 \phi' \phi^{-1} + \rho + \frac{r}{2} e_1^2 \sigma + 3\beta e_1 \sqrt{\frac{r}{2}} - e_1 \sqrt{\frac{r}{2}} \frac{\phi''}{\phi}, \\ u &= -\frac{\phi'}{r} \phi^{-1} + \frac{1}{2r} \frac{\phi''}{\phi} - \frac{\sigma e_1}{\sqrt{2r}} + \frac{\sigma}{r} - \frac{\beta}{2r}. \end{aligned} \right\} \tag{26}$$

式(26)中: ϕ 已由上面解出.

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Painlevé Analysis and Explicit Solutions for
a Generalized Lorenz System

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Abstract: A four-dimensional generalized Lorenz system with the Hamiltonian function is considered. The system is studied by Painlevé analysis method. The singular manifold expansion is finite “truncation” by means of resonances, and it is proved that the system is Painlevé integrability. The self-Bäcklund transformation of the system is gotten out. Some explicit solutions are obtained by means of the Schwarz derivative equation.

Keywords: generalized Lorenz system; Painlevé analysis; resonances; Bäcklund transformation; Schwarz derivative

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