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# 解抛物型方程的一个高精度显式差分格式

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摘要: 引入耗散项的方法, 构造一个条件稳定的显格式, 其稳定性条件为  $r \leq 1/2$ , 截断误差可达到  $O(\tau^2 + h^4 + \frac{\tau^2}{h^2})$ . 当  $\tau = O(h)$  时, 此格式可逼近精度, 特别当  $\tau = O(h^2)$  时, 格式达到二阶精度. 数值例子表明, 所建立的差分格式是有效的.

关键词: 二阶抛物型方程; 高精度; 差分格式; 耗散项

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在渗流、扩散、热传导等领域中, 经常会遇到求解抛物型方程的问题. 其一维模型的初边值问题为

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = f(x), \quad u(0, t) = g_1(t), \quad u(L, t) = g_2(t). \quad (1)$$

式(1)中,  $0 < x < L, t > 0$  对于此初边值问题, 古典隐格式<sup>[1]</sup>截断误差为  $O(\tau + h^2)$ , 古典显格式<sup>[2]</sup>和文[3]的截断误差为  $O(\tau + h^2)$ , 且稳定性条件为  $r \leq 1/2$  文[4]构造了此方程的一族含双参数的差分格式. 本文构造一个加耗散项的三层显式差分格式, 截断误差可达  $O(\tau^2 + h^4 + \frac{\tau^2}{h^2})$ .

## 1 差分格式的构造

现分别用  $\tau, h$  表示时间  $t$  及空间  $x$  方向的步长, 设  $r = \tau/h^2$  为步长比, 用  $u_j^n$  表示  $u(x, t)$  在节点  $(x_j, t_n)$  处的近似值<sup>[2]</sup>, 并引入耗散项  $-\varepsilon \frac{\tau^2}{h^2} \frac{\delta^2 u_j^n}{\tau^2}$ , 构造的 3 层含参数差分格式为

$$\begin{aligned} & \left( \frac{1}{2} + \alpha \right) \frac{u_j^{n+1} - u_j^n}{\tau} + \frac{1}{12} \left( \frac{u_{j+1}^n - u_{j-1}^n}{\tau} + \frac{u_{j-1}^n - u_{j-2}^n}{\tau} \right) + \left( \frac{1}{3} - \alpha \right) \frac{u_j^n - u_j^{n-1}}{\tau} = \\ & (1 + \alpha) \frac{\delta_x^2 u_j^n}{h^2} - \alpha \frac{\delta_x^2 u_j^{n-1}}{h^2} - \varepsilon \frac{\tau^2}{h^2} \frac{\delta^2 u_j^n}{\tau^2}. \end{aligned} \quad (2)$$

式(2)中,  $\delta_x^2 u_j^n$  和  $\delta_t^2 u_j^n$  分别表示关于空间  $x$  和时间  $t$  方向的中心差分, 即  $\delta_x^2 u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$ ,  $\delta_t^2 u_j^n = u_j^{n+1} - 2u_j^n + u_j^{n-1}$ .

## 2 截断误差的讨论

当初边值问题(1)的解  $u(x, t)$  充分光滑时, 如下关系式成立. 有

$$\frac{\partial^{p+q} u}{\partial t^p \partial x^q} = \frac{\partial^{2p+q} u}{\partial x^{2p+q}}, \quad p, q = 0, 1, 2, \dots \quad (3)$$

记  $\Delta u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}$ ,  $\nabla u_j^n = \frac{u_j^n - u_j^{n-1}}{\tau}$ . 把式(2)中各节点上的  $u$  在节点  $(x_j, t_n)$  处进行 Taylor 展开, 可得

$$\Delta u(x_j, t_n) = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\tau} = \frac{\partial u}{\partial t} + \frac{\tau}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3} + \frac{\tau^3}{24} \frac{\partial^4 u}{\partial t^4} + O(\tau^4),$$

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$$\begin{aligned}\nabla u(x_j, t_n) &= \frac{u(x_j, t_n) - u(x_j, t_{n-1})}{\tau} = \frac{\partial u}{\partial t} - \frac{\tau}{2} \frac{\partial^2 u}{\partial t^2} + \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3} - \frac{\tau^3}{24} \frac{\partial^4 u}{\partial t^4} + O(\tau^4), \\ \nabla u(x_{j+1}, t_n) + \nabla u(x_{j-1}, t_n) &= 2 \frac{\partial u}{\partial t} - \tau \frac{\partial^2 u}{\partial t^2} + h^2 \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\tau^2}{3} \frac{\partial^3 u}{\partial t^3} - \frac{\tau^3}{12} \frac{\partial^4 u}{\partial x^4 \partial t} - \\ &\quad \frac{\tau h^2}{2} \frac{\partial^4 u}{\partial x^4 \partial t^2} + \frac{\tau^2 h^2}{6} \frac{\partial^5 u}{\partial x^2 \partial t^3} + \frac{h^4}{12} \frac{\partial^5 u}{\partial x^4 \partial t} - \frac{\tau h^4}{28} \frac{\partial^6 u}{\partial x^4 \partial t^2} + O(\tau^4 + h^6), \\ \frac{\delta_x^2 u(x_j, t_n)}{h^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4} + \frac{h^4}{360} \frac{\partial^6 u}{\partial x^6} + O(h^6), \\ \frac{\delta_x^2 u(x_j, t_{n-1})}{h^2} &= \frac{\partial^2 u}{\partial x^2} - \tau \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4} + \frac{\tau^2}{2} \frac{\partial^4 u}{\partial x^2 \partial t^2} - \frac{\tau^3}{6} \frac{\partial^5 u}{\partial x^2 \partial t^3} - \frac{\tau h^2}{12} \frac{\partial^5 u}{\partial x^4 \partial t} + \\ &\quad \frac{h^4}{360} \frac{\partial^6 u}{\partial x^6} + \frac{\tau h^2}{24} \frac{\partial^6 u}{\partial x^4 \partial t^2} - \frac{\tau h^4}{360} \frac{\partial^7 u}{\partial x^4 \partial t^3} + O(\tau^4 + h^6), \\ \frac{\delta_t^2 u(x_j, t_n)}{\tau^2} &= \frac{\partial^2 u}{\partial t^2} + O(\tau^2).\end{aligned}$$

将上述各式代入式(2), 可得

$$\begin{aligned}\frac{\partial u}{\partial t} + \alpha \tau \frac{\partial^2 u}{\partial t^2} + \frac{h^2}{12} \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3} + \frac{\tau h^2}{24} \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{h^4}{144} \frac{\partial^5 u}{\partial x^4 \partial t} + O(\tau^3 + h^6) = \\ \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\tau^2}{h^2} \frac{\partial^2 u}{\partial t^2} + \alpha \tau \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4} - \frac{\alpha^2}{2} \tau^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\alpha}{12} \tau h^2 \frac{\partial^5 u}{\partial x^4 \partial t} + \frac{h^4}{360} \frac{\partial^6 u}{\partial x^6} + O(\tau^3 + h^6).\end{aligned}$$

由关系式(3), 可得(2)的局部截断误差为

$$R_j^n = \left(\frac{1}{6} + \frac{\alpha}{2}\right) \tau^2 \frac{\partial^3 u}{\partial t^3} - \left(\frac{1}{24} + \frac{\alpha}{12}\right) \tau h^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{h^4}{240} \frac{\partial^5 u}{\partial x^4 \partial t} + \varepsilon \frac{\tau^2}{h^2} \frac{\partial^2 u}{\partial t^2} O(\tau^3 + h^6).$$

故格式(2)的局部截断误差阶为  $O(\tau^2 + h^4 + \frac{\tau^2}{h^2})$ . 此格式当  $\tau = O(h)$  时才有逼近精度, 若令  $\tau = O(h^2)$ , 则格式可达二阶精度.

### 3 差分格式的稳定性

引理 即 Miller 准则<sup>[5]</sup>. 设  $A > 0$ , 实系数二次方程  $Ax^2 + Bx + C = 0$  的两根按模小于等于 1 的充要条件是  $A - C \geq 0, A - B + C \geq 0, A + B + C \geq 0$ .

定理 1 显式格式(2)稳定的一个充分条件:  $1/6 \leq \alpha \leq 1/12r, \varepsilon \geq 1$ . 其中,  $1/6 \leq \alpha \leq 1/12r$ , 隐含条件  $r \leq 1/2$ .

证明 由 Fourier 分析法<sup>[6]</sup>可知, 格式(2)的特征方程为

$$A\lambda^2 + B\lambda + C = 0. \quad (4)$$

式(4)中,  $A = \frac{1}{3} + \alpha + \varepsilon, B = \frac{1}{3} - \frac{1}{3}s^2 - 2\alpha + 4rs^2 + 4\alpha s^2 - 2\varepsilon, C = \frac{1}{3}s^2 - \frac{2}{3} + \alpha - 4\alpha s^2 + \varepsilon$ . 其中,  $s = \sin \frac{\alpha}{2}$ . 当  $\frac{1}{6} \leq \alpha \leq \frac{1}{12r}, \varepsilon \geq 1$  时,  $A > 0, A - C = 1 - \frac{1}{3}s^2 + 4\alpha s^2 \geq \frac{2}{3} + 4\alpha s^2 > 0, A - B + C = 4\alpha + 4\varepsilon + \frac{2}{3}s^2 - 8\alpha s^2 - 4rs^2 - \frac{2}{3} \geq 4\alpha + 4\varepsilon s + \frac{2}{3}s^2 - 8\alpha s^2 - 4r - \frac{2}{3} = (4\alpha - \frac{2}{3}) + 4r(\varepsilon - 1) + (\frac{2}{3} - 8\alpha)s^2 \geq 0, A + B + C = 4rs^2 \geq 0$ .

由引理可知, 特征方程(4)的特征根的模小于等于 1. 又因为  $A - C > 0$ , 由韦达定理可知特征方程(4)无重根, 即当  $1/6 \leq \alpha \leq 1/12r, \varepsilon \geq 1$  时, 格式(2)稳定. 定理得证.

### 4 数值例子

解二阶抛物型方程初边值问题

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \sin x, \quad u(0, t) = 0, \quad u(\pi, t) = 0. \quad (5)$$

式(5)中,  $0 < x < \pi, t > 0$  取  $h = \pi/15, \tau = rh^2$ . 为简便计, 用精确解  $u(x, t) = e^{-t} \sin x$  计算第 1 层的值.

将格式(2)在  $t=10$  时的数值解与精确解作比较, 结果如表 1 所示.

由表 1 可以看出, 定理 1 的稳定性条件  $1/6 \leq \alpha \leq 1/12r$  和  $\varepsilon \geq 1$ , 仅仅是格式(2)的一个充分条件, 不同的参数对格式稳定性的影响不同. 从表 1 还可看出, 其最后一行为差分格式(2)发散时的数据. 结果表明, 理论分析与实际计算相符合.

表 1 格式(2)数值解与精确解的比较  
Tab. 1 The numerical solution in comparison with the real solution

项目	$r$	$\alpha$	$\varepsilon$	$x$			
				$2\pi/15$	$4\pi/15$	$6\pi/15$	$8\pi/15$
精确解	—	—	—	$1.846 \times 10^{-5}$	$3.373 \times 10^{-5}$	$4.317 \times 10^{-5}$	$4.515 \times 10^{-5}$
差分解	1/2	1/6	1.0	$1.710 \times 10^{-5}$	$3.125 \times 10^{-5}$	$4.000 \times 10^{-5}$	4.183
差分解	1/4	1/6	1.0	$1.828 \times 10^{-5}$	$3.341 \times 10^{-5}$	$4.276 \times 10^{-5}$	$4.472 \times 10^{-5}$
差分解	1/4	1/3	0.7	$1.846 \times 10^{-5}$	$3.370 \times 10^{-5}$	$4.313 \times 10^{-5}$	$4.510 \times 10^{-5}$
差分解	1/10	1/6	1.6	$1.846 \times 10^{-5}$	$3.374 \times 10^{-5}$	$4.318 \times 10^{-5}$	$4.516 \times 10^{-5}$
差分解	1/10	5/6	1.7	$1.846 \times 10^{-5}$	$3.372 \times 10^{-5}$	$4.316 \times 10^{-5}$	$4.513 \times 10^{-5}$
差分解	1.0	1/6	1.6	$2.351 \times 10^{19}$	$4.296 \times 10^{19}$	$5.499 \times 10^{19}$	$5.750 \times 10^{19}$

参考文献:

[1] 苏煜城, 吴启光. 偏微分方程数值解法[M]. 北京: 气象出版社, 1989: 47, 58-60.  
[2] 余德浩, 汤华中. 偏微分方程数值解法[M]. 北京: 科学出版社, 2004: 106-109.  
[3] 杨情民. 解抛物型方程的一族显格式[J]. 高等学校计算数学学报, 1981(4): 306-317.  
[3] 曾文平. 抛物型方程的一族双参数高精度恒稳差分格式[J]. 华侨大学学报: 自然科学版, 2002, 23(4): 327-331.  
[5] MILLER J J H. On the location of zeros of certain classes of polynomials with application to numerical analysis[J]. J Inst Math Appls, 1971, 8(3): 394-406.  
[6] RICHTMYER R D, MORTON K W. Difference method for initial value problems[M]. 2nd ed. New York: Wiley, 1967.

An Explicit Difference Scheme with High Order Accuracy for Solving Parabolic Equation

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**Abstract:** This paper gives a three layer explicit difference scheme with the dissipative term. Its local truncation error is in the order of  $O(\tau^2 + h^4 + \frac{\tau^2}{h^2})$ . Numerical examples shown that it is effective and practice consistent with theoretical analysis.

**Keywords:** second order parabolic equation; high accuracy; difference scheme; dissipative term

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