

强迫拟线性波动方程的初值问题

陈安全 郑永树

(华侨大学数学系, 福建 泉州 362021)

摘要 研究具非线性耗散项的强迫拟线性波动方程的初值问题. 对初值的 C^0 模不加小性限制, 而需其一阶导数的 C^0 模足够小. 利用几个关键的先验估计, 证明初值问题的整体光滑解的存在性.

关键词 非线性耗散项, 强迫拟线性波动方程, 先验估计, 整体光滑解

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1 问题和假设

文[1~3]研究了具耗散项的强迫拟线性波动方程的初值问题的整体光滑解. 本文考虑具非线性耗散项的强迫拟线性波动方程的初值问题. 即

$$u_t - (\Phi(u_x))_x + f(u_t) = g(t, x), \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}, \quad (1)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbf{R} \quad (2)$$

令 $u_x = z$, $u_t = \omega$, 则初值问题(1), (2)可改写为等价问题. 有

$$\omega - \Phi(z)_x + f(\omega) = g(t, x), \quad z_t - \omega_x = 0, \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}, \quad (3)$$

$$\omega(0, x) = \omega_0(x), \quad z(0, x) = z_0(x), \quad x \in \mathbf{R} \quad (4)$$

在上式中, $\omega_0(x) = u_1(x)$, $z_0(x) = u'_0(x)$. 文[3]在对初值的 C^0 模的小性限制下, 证明了初值问题(1), (2)整体光滑解的存在唯一性. 本文则解除对初值的模的小性限制, 而对其一阶导数作了小性假设. 假设初值问题(1), (2)满足 4 个条件. (A₁) $\Phi(z) \in C^2(\mathbf{R})$, $\Phi'(z) > 0$, $\forall z \in \mathbf{R}$, $\lim_{z \rightarrow \pm\infty} \int_0^z \sqrt{\Phi'(\tau)} d\tau = \pm\infty$.

(A₂) $f(\omega) \in C^1(\mathbf{R})$, $f(0) = 0$, $f'(\omega) \geq 2\alpha$, $\alpha > 0$ 为常数. (A₃) $g, g_x \in C([0, \infty) \times \mathbf{R})$, $G(t) \sup_x |g(t, x)| \in L^1(0, \infty)$, $g_1(t) = \sup_x |g_x(t, x)| \in L^1(0, \infty)$. (A₄) $u_0(x) \in C_b^2(\mathbf{R})$, $u_1(x) \in C_b^1(\mathbf{R})$, 即 $z_0(x), \omega_0(x) \in C_b^1(\mathbf{R})$. 其中, $C_b^i(\mathbf{R})$ 表示 $C^i(\mathbf{R})$ 中的有界集, $i = 1, 2$. 经简单计算, 方程(1)有特征值, $\lambda = -\sqrt{\Phi'(z)}$, $\mu = \sqrt{\Phi'(z)}$. 相应的 Riemann 不变量是 $r = \omega + \Phi(z)$, $s = \omega - \Phi(z)$. 其中, $\Phi(z) = \int_0^z \sqrt{\Phi'(\tau)} d\tau$ 在经典解的意义下, 初值问题(3), (4)又可化为等价的初值问题. 即

$$r_t + \lambda_x = g(t, x) - f(r+s)/2, \quad r_t + \mu_x = g(t, x) - f(r+s)/2, \quad (5)$$

$$r(0, x) = r_0(x), \quad s(0, x) = s_0(x). \quad (6)$$

上两式中, $r_0(x) = u_1(x) + \Phi(u'_0(x))$, $s_0(x) = u_1(x) - \Phi(u'_0(x))$.

2 主要结果及其证明

2.1 主要结果

定理 假设条件(A₁), (A₂), (A₃)和(A₄)成立, 如果 $\sup_x |r'_0(x)| + \sup_x |s'_0(x)| + \|g_1(t)\|_{L^1(0, \infty)}$

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作者简介 陈安全(1975-), 男, 硕士研究生, 现为厦门理工学院(福建 厦门 361000)讲师, 主要从事经济统计和偏微分方程的研究; 通信作者: 郑永树(1939-), 男, 教授, E-mail: zys@hqu.edu.cn

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足够小, 则初值问题(1), (2)(等价于问题(5), (6))在上半平面($t \geq 0$)存在唯一的整体光滑解.

2.2 解的模估计

引理 1 在假设条件(A_1), (A_2), (A_3)和(A_4)下, 初值问题(5), (6)在光滑解存在区域内成立, 有

$$|r(t, x)| \leq M_0 + \int_0^\infty G(\tau) d\tau \quad |s(t, x)| \leq M_0 + \int_0^\infty G(\tau) d\tau \quad (7)$$

在式(7)中, $M_0 = \max\{\sup_x |r_0(x)|, \sup_x |s_0(x)|\}$.

证明 由条件(A_2), 方程组(5)可改写为

$$\frac{dr}{dt} = g(t, x) - \frac{1}{2}f'(\theta\omega)(r+s), \quad \frac{ds}{dt} = g(t, x) - \frac{1}{2}f'(\theta\omega)(r+s). \quad (8)$$

在式(8)中, $0 < \theta = \theta(r, s) < 1, f'(\omega) = \frac{df(\omega)}{d\omega}, \frac{d}{dt} = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}, \frac{d}{dt} = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}$. 令 $F_1(t) = \sup_x f'(\omega(t, x)), r^- = re^{\frac{1}{2}\int_0^t F_1(\tau) d\tau}, \bar{s} = se^{\frac{1}{2}\int_0^t F_1(\tau) d\tau}$. 于是, 由此可得到

$$\left. \begin{aligned} \frac{dr^-}{dt} &= g(t, x)e^{\frac{1}{2}\int_0^t F_1(\tau) d\tau} + \frac{1}{2}(F_1(t) - f'(\theta\omega))r^- - \frac{1}{2}f'(\theta\omega)\bar{s}, \\ \frac{d\bar{s}}{dt} &= g(t, x)e^{\frac{1}{2}\int_0^t F_1(\tau) d\tau} + \frac{1}{2}(F_1(t) - f'(\theta\omega))\bar{s} - \frac{1}{2}f'(\theta\omega)r. \end{aligned} \right\} \quad (9)$$

又令 $h(t) = \max\{\sup_x |r^-(t, x)|, \sup_x |\bar{s}(t, x)|\}$. 设 (t, x) 为 C^1 解存在区域内的任意一点, 并过点 (t, x) 作 λ 特征线和 μ 特征线, 分别交于 x 轴的点 $(0, \alpha)$ 和 $(0, \beta)$, 沿特征线从 0 到 t 分别积分式(9). 注意到 M_0 和 $f'(\omega) \geq 2\omega > 0, F_1 - f'(\theta\omega) \geq 0$, 则有

$$\left. \begin{aligned} |r^-(t, x)| &\leq M_0 + \frac{1}{2} \int_0^t F_1(\tau) h(\tau) d\tau + \int_0^t |g| e^{\frac{1}{2}\int_0^\tau F_1(\tau) d\tau} d\tau, \\ |\bar{s}(t, x)| &\leq M_0 + \frac{1}{2} \int_0^t F_1(\tau) h(\tau) d\tau + \int_0^t |g| e^{\frac{1}{2}\int_0^\tau F_1(\tau) d\tau} d\tau. \end{aligned} \right\} \quad (10)$$

进而由假设条件(A_3), $h(t)$ 的条件和式(10), 可得到

$$\left. \begin{aligned} h(t) &\leq M_0 e^{\frac{1}{2}\int_0^t F_1(\tau) d\tau} + e^{\frac{1}{2}\int_0^t F_1(\tau) d\tau} \int_0^t G(\tau) d\tau, \\ |r(t, x)| &\leq M_0 + \int_0^\infty G(\tau) d\tau, \quad |s(t, x)| \leq M_0 + \int_0^\infty G(\tau) d\tau \end{aligned} \right\} \quad (11)$$

引理 1 证毕.

推论 在引理 1 的假设之下, 初值问题(3), (4)在光滑解存在区域内成立. 则 $|\omega(t, x)| \leq \bar{M}_0, z \leq \bar{z}(t, x) \leq \bar{z}$. 其中, $\bar{M}_0 = M_0 + \int_0^\infty G(\tau) d\tau, \bar{M}_0 = \int_0^\infty \sqrt{G(\tau)} d\tau, \bar{M}_0 = \int_0^\infty \sqrt{G(\tau)} d\tau$.

2.3 解的偏导数估计

引理 2 在引理 1 的假设之下, 如果 $M_1 = \max\{\sup_x |r'_0(x)|, \sup_x |s'_0(x)|\} + \|g_1(t)\|_{L^1(0, \infty)}$ 足够小, 则初值问题(5), (6)在光滑解的存在区域内成立. 即 $|r_x(t, x)| \leq D_1, |s_x(t, x)| \leq D_1$. 其中, $D_1 = \frac{\mu_M M_1}{\mu_m}, \mu_M = \max_{z \leq \bar{z}} \mu(z), \mu_m = \min_{z \leq \bar{z}} \mu(z)$.

证明 首先, 通过直接计算, 由式可得到

$$\left. \begin{aligned} \frac{d\mu_x}{dt} &= -\mu_x \mu_x^2 - \mu_x \mu_x s_x - \frac{1}{2}f'(\omega) \mu_x - \frac{1}{2}f'(\omega) \mu_s + \mu_g x, \\ \frac{d\mu_s}{dt} &= -\mu_x \mu_s^2 - \mu_x \mu_x s_x - \frac{1}{2}f'(\omega) \mu_x - \frac{1}{2}f'(\omega) \mu_s + \mu_g x. \end{aligned} \right\} \quad (12)$$

令 $U = \mu_x, V = -\mu_s, A = \frac{1}{2}f'(\omega) + \mu_x, B = \frac{1}{2}f'(\omega) + \mu_s$. 则有

$$\frac{dU}{dt} = A(V - U) + \mu_g x, \quad \frac{dV}{dt} = B(U - V) - \mu_g x. \quad (13)$$

由式(13)可得

$$\frac{dU}{dt} \leq A(V - U) + \mu_g g_1(t), \quad \frac{dV}{dt} \leq B(U - V) + \mu_g g_1(t). \quad (14)$$

又令 $\overline{U} = U - \mu_M \int_0^1 g_1(\tau) d\tau$, $\overline{V} = V - \mu_M \int_0^1 g_1(\tau) d\tau$ 则有

$$\frac{d\overline{U}}{dt} \leq A(\overline{V} - \overline{U}), \quad \frac{d\overline{V}}{dt} \leq B(\overline{U} - \overline{V}). \quad (15)$$

先验假设 $|\mu_{rx}| \leq \frac{\alpha}{2}$, $|\mu_{sx}| \leq \frac{\alpha}{2}$. 根据假设条件 (A_2) 及先验假设, 可得到 $A \geq \frac{\alpha}{2} > 0$, $B \geq \frac{\alpha}{2} > 0$.

根据文[4, 5]的极值原理和常微分方程的比较定理, 可得到

$$\overline{U}(t, x), \overline{V}(t, x) \leq \max\{\sup_x |\overline{U}(0, x)|, \sup_x |\overline{V}(0, x)|\} \leq \mu_M \max(\sup_x |r'_0(x)|, \sup_x |s'_0(x)|).$$

从而有

$$\left. \begin{aligned} (\mu_{rx})(t, x) &\leq \mu_M \max(\sup_x |r'_0(x)|, \sup_x |s'_0(x)|) + \mu_M \int_0^1 g_1(\tau) d\tau \\ - (\mu_{sx})(t, x) &\leq \mu_M \max(\sup_x |r'_0(x)|, \sup_x |s'_0(x)|) + \mu_M \int_0^1 g_1(\tau) d\tau \end{aligned} \right\} \quad (16)$$

另一方面, 由式(13)可得到

$$\frac{d\overline{U}}{dt} \geq A(\overline{V} - \overline{U}) - \mu_M g_1(t), \quad \frac{d\overline{V}}{dt} \geq B(\overline{U} - \overline{V}) - \mu_M g_1(t).$$

类似地, 可得到

$$\left. \begin{aligned} (\mu_{rx})(t, x) &\geq -\mu_M \max(\sup_x |r'_0(x)|, \sup_x |s'_0(x)|) - \mu_M \int_0^1 g_1(\tau) d\tau \\ - (\mu_{sx})(t, x) &\geq -\mu_M \max(\sup_x |r'_0(x)|, \sup_x |s'_0(x)|) - \mu_M \int_0^1 g_1(\tau) d\tau \end{aligned} \right\} \quad (17)$$

由式(16), (17), 可得到

$$|r_x(t, x)|, |s_x(t, x)| \leq (\mu_M M_1) / \mu_m = D_1. \quad (18)$$

即证先验假设是合理的. 事实上, 令 $K = \frac{1}{4} \max \left| \frac{\sigma''(z)}{(\sigma'(z))^{3/2}} \right|$. 由式(16), 可得到

$$|(\mu_{rx})(t, x)| = \left| \frac{\sigma''(z)}{4(\sigma'(z))^{3/2}} \mu_{rx}(t, x) \right| \leq K \mu_M M_1. \quad (19)$$

因此, 只要 M_1 足够小, 则 $K \mu_M M_1 \leq \frac{\alpha}{2}$. 所以 $|(\mu_{rx})(t, x)| \leq \frac{\alpha}{2}$. 类似可验证 $|(\mu_{sx})(t, x)| \leq \frac{\alpha}{2}$. 引理 2

证毕. 根据经典解的局部存在性定理和引理 1, 2, 可以证明定理成立.

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The Initial Value Problems of Forced Quasilinear Wave Equation

Chen Anquan Zheng Yongshu

(Department of Mathematics, Huaqiao University, 362021, Quanzhou, China)

Abstract We consider the initial value problems of forced quasilinear wave equation with nonlinear dissipation. Not requiring that the C^0 norm of the initial data is small but the C^0 norm of the first order derivative is sufficiently small, and using some prior estimates, we prove the existence of the global smooth solutions for the initial value problem.

Keywords nonlinear dissipation, forced quasilinear wave equations, priori estimate, globally smooth solution