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梁振动方程的多辛 Fourier 拟谱算法

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摘要　利用 Fourier 拟谱方法, 分别对梁振动方程的辛格式进行空间和时间方向上的离散, 得到相应的多辛守恒律. 文中证明了离散局部能量守恒, 并用实例说明理论分析是正确的.

关键词　梁振动方程, 多辛, Fourier 拟谱方法, 守恒律

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考虑等截面梁横向自由振动方程的周期初值问题, 即

$$\left.\begin{aligned}&\frac{\partial^2 u}{\partial t^2}+a^2\frac{\partial^4 u}{\partial x^4}=0,\qquad u(x+2\pi,t)=u(x,t),\qquad (x,t)\in\mathbf{R}\times I,\\&u(x,0)=f_0(x),\qquad u_t(x,0)=f_1(x),\qquad x\in\mathbf{R}.\end{aligned}\right\}\tag{1}$$

在式(1)中, $a^2=\dfrac{EJ}{\rho A}$, ρ为单位面积梁的质量, A 为梁横截面的面积, E 为材料弹性模量, J 为截面对中性轴的惯性矩, EJ 为抗弯刚度, 均为常数. 引入正则动量[1] $v=u_x$, $u_t=p_x$, $w_x=p$, 可得到方程组(1)的多辛 Hamilton 方程组为

$$w_t+a^2v_x=0,\qquad -a^2u_x=-a^2v,\qquad -u_t+p_x=0,\qquad -w_x=-p.\tag{2}$$

令状态变量 $\boldsymbol{Z}=(u,v,w,p)^{\mathrm{T}}$, 则 Hamilton 方程组(2)可以表示为

$$\boldsymbol{M}\partial_t\boldsymbol{Z}+\boldsymbol{K}\partial_x\boldsymbol{Z}=\nabla_z S(\boldsymbol{Z}).\tag{3}$$

在式(3)中, $S(\boldsymbol{Z})=-\dfrac{1}{2}(a^2v^2+p^2)$, $\boldsymbol{M}=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix}$, $\boldsymbol{K}=\begin{pmatrix}0&a^2&0&0\\a^2&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix}$. 文[2~5]将 Fourier 拟谱离散应用于多辛 Hamilton 系统, 并对一些问题进行了数值计算. 本文应用拟谱离散方法对多辛 Hamilton 方程组(2)进行了研究.

1　梁振动方程的多辛 Fourier 拟谱离散

以 $I_m\boldsymbol{Z}(x,t)$表示 $\boldsymbol{Z}(x,t)$在配置点 $x_j=\dfrac{\pi}{m}\cdot j$, $(j=0,1,2,\cdots,2m-1)$处的插值近似, 则有[4]

$$I_m\boldsymbol{Z}(x,t)=\sum_{l=-m}^{m}\hat{z}_l\cdot\mathrm{e}^{ilx}.$$

上式中, $\hat{z}_l=\dfrac{1}{2mc_l}\cdot\sum\limits_{j=0}^{2m-1}z_j\mathrm{e}^{-ilx_j}$, $c_l=1(|l|\neq m)$, $c_{-m}=c_m=2$, $z_j=z(x_j,t)$. 在配置点处 $I_m\boldsymbol{Z}(x_j,t)=z_j$, 则可得$\dfrac{\partial I_m\boldsymbol{Z}(x_j,t)}{\partial x}=(\boldsymbol{D}_1\boldsymbol{Z})_j$. 其中, $\boldsymbol{Z}=(z_0,z_1,\cdots,z_{2m-1})^{\mathrm{T}}$, $(\boldsymbol{D}_1)_{j,n}=\begin{cases}\dfrac{1}{2}^{j+n}\cdot\cot(\mu\dfrac{x_jx_n}{2}), & j\neq n,\\0, & j=n.\end{cases}$

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记 $\boldsymbol{U}=(u_0, u_1, \cdots, u_{2m-1})^{\mathrm{T}}$, $\boldsymbol{V}=(v_0, v_1, \cdots, v_{2m-1})^{\mathrm{T}}$, $\boldsymbol{W}=(w_0, w_1, \cdots, w_{2m-1})^{\mathrm{T}}$, $\boldsymbol{P}=(p_0, p_1, \cdots, p_{2m-1})^{\mathrm{T}}$. 则方程组(2)的 Fourier 半拟谱离散格式为

$$\left.\begin{aligned} &\frac{\mathrm{d} w_j}{\mathrm{d} t}+a^2(\boldsymbol{D}_1 \boldsymbol{V})_j=0, \\ &-a^2(\boldsymbol{D}_1 \boldsymbol{U})_j=-a^2 v_j, \\ &\frac{\mathrm{d} u_j}{\mathrm{d} t}+(\boldsymbol{D}_1 \boldsymbol{P})_j=0, \\ &-(\boldsymbol{D}_1 \boldsymbol{W})_j=-p_j, \end{aligned}\right\} \tag{4}$$

全拟谱离散格式为

$$\left.\begin{aligned} &\frac{w_j^{n+1}-w_j^n}{\Delta t}+a^2(\boldsymbol{D}_1 \boldsymbol{V}^{n+\frac{1}{2}})_j=0, \\ &-a^2(\boldsymbol{D}_1 \boldsymbol{U}^{n+\frac{1}{2}})_j=-a^2 v_j^{n+\frac{1}{2}}, \\ &-\frac{u_j^{n+1}-u_j^n}{\Delta t}+(\boldsymbol{D}_1 \boldsymbol{P}^{n+\frac{1}{2}})_j=0, \\ &-(\boldsymbol{D}_1 \boldsymbol{W}^{n+\frac{1}{2}})_j=-p_j^{n+\frac{1}{2}}. \end{aligned}\right\} \tag{5}$$

在方程(4), (5) 中, $j=0, 1, 2, \cdots, 2m-1$.

2　局部能量守恒及算法的稳定性

定理 **1**　半离散拟谱格式(4)具有 $2m$ 个半离散的多辛守恒律

$$\frac{\mathrm{d} t}{\mathrm{d} t} \cdot \omega_j+\sum_{k=0}^{2m-1}(\boldsymbol{D}_1)_{j, k} \cdot \kappa_{j, k}=0, \tag{6}$$

其中, $\omega_j=\frac{1}{2}(\mathrm{d} z_j \wedge \boldsymbol{M} \mathrm{d} z_j)$, $\kappa_{j, k}=\frac{1}{2}(\mathrm{d} z_j \wedge \boldsymbol{K} \mathrm{d} z_k+\mathrm{d} z_k \wedge \boldsymbol{K} \mathrm{d} z_j)$, $(j=0, 1, 2, \cdots, 2m-1)$.

证明　格式(4) 的矩阵形式为

$$\boldsymbol{M} \frac{\mathrm{d} z_j}{\mathrm{d} t}+\boldsymbol{K} \sum_{k=0}^{2m-1}(\boldsymbol{D}_1)_{j, k} \cdot z_k=\nabla_z S(z_j), \qquad j=0, 1, 2, \cdots, 2m-1,$$

用 $\mathrm{d} z_j$ 对上式的变分形式作外积, 可得定理 1 的证明.

定理 **2**　全离散格式(5)满足离散局部能量守恒律

$$\frac{\omega_j^{n+1}-\omega_j^n}{\Delta t}+\sum_{k=0}^{2m-1}(\boldsymbol{D}_1)_{j, k} \cdot \kappa_{j, k}^{n+\frac{1}{2}}=0,$$

以及整体守恒性质

$$\sum_{j=0}^{2m-1} \omega_j^{n+1}=\sum_{j=0}^{2m-1} \omega_j^n.$$

其中, $\omega_j^n=\frac{1}{2}(\mathrm{d} z_j^n \wedge \boldsymbol{M} \mathrm{d} z_j^n)$, $\kappa_{j, k}^{n+\frac{1}{2}}=(\mathrm{d} z_j^{n+\frac{1}{2}} \wedge \boldsymbol{K} \mathrm{d} z_k^{n+\frac{1}{2}})+\mathrm{d} z_k^{n+\frac{1}{2}} \wedge \boldsymbol{K} \mathrm{d} z_j^{n+\frac{1}{2}})$, $(j=0, 1, 2, \cdots, 2m-1)$.

证明　全离散格式(5)的矩阵变分形式为

$$\boldsymbol{M} \frac{(\mathrm{d} z_j^{n+1}-\mathrm{d} z_j^n)}{\Delta t}+\boldsymbol{K} \sum_{k=0}^{2m-1}(\boldsymbol{D}_1)_{j, k} \mathrm{d} z_j^{n+\frac{1}{2}}=S_{zz}(z_k^{n+\frac{1}{2}})) \mathrm{d} z_j^{n+\frac{1}{2}}. \tag{7}$$

注意到 $\mathrm{d} z_j^{n+\frac{1}{2}} \wedge S_{zz}(z_j^{n+\frac{1}{2}}) \mathrm{d} z_j^{n+\frac{1}{2}}=0$, 用 $\mathrm{d} z_j^{n+\frac{1}{2}}$ 与式(7)作外积, 可得定理 2 的证明.

全离散格式(5) 的向量形式为

$$\left.\begin{aligned} &\frac{\boldsymbol{W}^{n+1}-\boldsymbol{W}^n}{\Delta t}+a^2(\boldsymbol{D}_1 \boldsymbol{V}^{n+\frac{1}{2}})=0, \\ &-a^2 \boldsymbol{D}_1 \boldsymbol{U}^{n+\frac{1}{2}}=-a^2 \boldsymbol{V}^{n+\frac{1}{2}}, \\ &-\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\Delta t}+\boldsymbol{D}_1 \boldsymbol{P}^{n+\frac{1}{2}}=0, \\ &-\boldsymbol{D}_1 \boldsymbol{W}^{n+\frac{1}{2}}=-\boldsymbol{P}^{n+\frac{1}{2}}. \end{aligned}\right\} \tag{8}$$

定理 **3**　格式(8) 满足离散局部能量守恒律, 即

$$\frac{E_j^{n+1}-E_j^n}{\Delta t}+(\boldsymbol{D}_1\boldsymbol{F}^{n+\frac{1}{2}})_j=0.$$

在上式中, $E_j^n=-\frac{1}{2}[a^2u_j^n(\boldsymbol{D}_1\boldsymbol{V}^n)_j+w_j^n(\boldsymbol{D}_1\boldsymbol{P}^n)_j]$, $(\boldsymbol{D}_1\boldsymbol{F}^{n+1})_j=\frac{2}{4\Delta t}\{a^2[u_j^n(\boldsymbol{D}_1\boldsymbol{V}^{n+1})_j-u_j^{n+1}(\boldsymbol{D}_1\boldsymbol{V}^n)_j]+[w_j^n(\boldsymbol{D}_1\boldsymbol{P}^{n+1})_j-w_j^{n+1}(\boldsymbol{D}_1\boldsymbol{P}^n)_j]\}$, $(j=0,1,2,\cdots,2m-1)$.

证明 $\boldsymbol{Z}=(u,v,w,p)^{\mathrm{T}}$, $\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{Z}_x=-a^2vu_x+a^2uv_x-pw_x+wp_x$, $S(\boldsymbol{Z})=-\frac{1}{2}(a^2v^2+p^2)$, $E=S(\boldsymbol{Z})-\frac{1}{2}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{Z}_x=-\frac{1}{2}(a^2v^2+p^2)-\frac{1}{2}(-a^2vu_x+a^2uv_x-pw_x+wp_x)=-\frac{1}{2}(a^2uv_x+wp_x)$, $E_j^n=-\frac{1}{2}[-a^2u_j^n(\boldsymbol{D}_1\boldsymbol{V}^n)_j+w_j^n(\boldsymbol{D}_1\boldsymbol{P}^n)_j]$, $E_j^{n+1}-E_j^n=\frac{1}{2}\{a^2[u_j^n(\boldsymbol{D}_1\boldsymbol{V}^n)_j-u_j^{n+1}(\boldsymbol{D}_1\boldsymbol{V}^{n+1})_j]+[w_j^n(\boldsymbol{D}_1\boldsymbol{P}^n)_j-w_j^{n+1}\cdot(\boldsymbol{D}_1\boldsymbol{P}^{n+1})_j]$. 又由于

$$F=\frac{1}{2}\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{Z}_t=\frac{1}{2}(-a^2vu_t+a^2uv_t-pw_t+wp_t),$$

$$\frac{\partial F}{\partial x}=\frac{1}{2}(-a^2v_tu_x+a^2uv_{tx}-p_xw_t+wp_{tx}),$$

因此有

$$(\boldsymbol{D}_1\boldsymbol{F}^{n+1})_j=\frac{2}{4\Delta t}\{a^2[u_j^n(\boldsymbol{D}_1\boldsymbol{V}^{n+1})_j-u_j^{n+1}(\boldsymbol{D}_1\boldsymbol{V}^n)_j]+[w_j^n(\boldsymbol{D}_1\boldsymbol{P}^{n+1})_j-w_j^{n+1}(\boldsymbol{D}_1\boldsymbol{P}^n)_j]\}.$$

由格式(8)可得

$$a^2(\boldsymbol{D}_1\boldsymbol{V}^{n+\frac{1}{2}})_j(u_j^{n+1}-u_j^n)+(\boldsymbol{D}_1\boldsymbol{P}^{n+\frac{1}{2}})_j(w_j^{n+1}-w_j^n)=0,$$

即

$$a^2(\boldsymbol{D}_1\boldsymbol{V}^{n+1}+\boldsymbol{D}_1\boldsymbol{V}^n)_j(u_j^{n+1}-u_j^n)+(\boldsymbol{D}_1\boldsymbol{P}^{n+1}+\boldsymbol{D}_1\boldsymbol{P}^n)_j(w_j^{n+1}-w_j^n)=0.$$

从而得

$$\begin{aligned}&a^2[u_j^n(\boldsymbol{D}_1\boldsymbol{V}^n)_j-u_j^{n+1}(\boldsymbol{D}_1\boldsymbol{V}^{n+1})_j]+[w_j^n(\boldsymbol{D}_1\boldsymbol{P}^n)_j-w_j^{n+1}(\boldsymbol{D}_1\boldsymbol{P}^{n+1})_j]=\\&a^2[u_j^{n+1}(\boldsymbol{D}_1\boldsymbol{V}^n)_j-u_j^n(\boldsymbol{D}_1\boldsymbol{V}^{n+1})_j]+[w_j^{n+1}(\boldsymbol{D}_1\boldsymbol{P}^n)_j-w_j^n(\boldsymbol{D}_1\boldsymbol{P}^{n+1})_j],\end{aligned}$$

即

$$\frac{E_j^{n+1}-E_j^n}{\Delta t}+(\boldsymbol{D}_1\boldsymbol{F}^{n+\frac{1}{2}})_j=0.$$

证毕.

从格式(8)中消去中间变量, 可得

$$\left.\begin{aligned}&\boldsymbol{U}^{n+1}=(1+a^2\Delta t^2\boldsymbol{D}_1^4)^{-1}(1-a^2\Delta t^2\boldsymbol{D}_1^4)\boldsymbol{U}^n+2\Delta t(1+a^2\Delta t^2\boldsymbol{D}_1^4)^{-1}\boldsymbol{D}_1^2W^n,\\&W^{n+1}=W^n-a^2\Delta t\boldsymbol{D}_1^2(\boldsymbol{U}^{n+1}+\boldsymbol{U}^n).\end{aligned}\right\}\tag{9}$$

上式中, 因 $\boldsymbol{D}_1^4$ 为对称矩阵, 且其特征值均非负, 因此, $(1+a^2\Delta t^2\boldsymbol{D}_1^4)^{-1}(1-a^2\Delta t^2\boldsymbol{D}_1^4)^{-1}$的谱半径恒小于1, 格式(9)绝对稳定.

3 数值试验

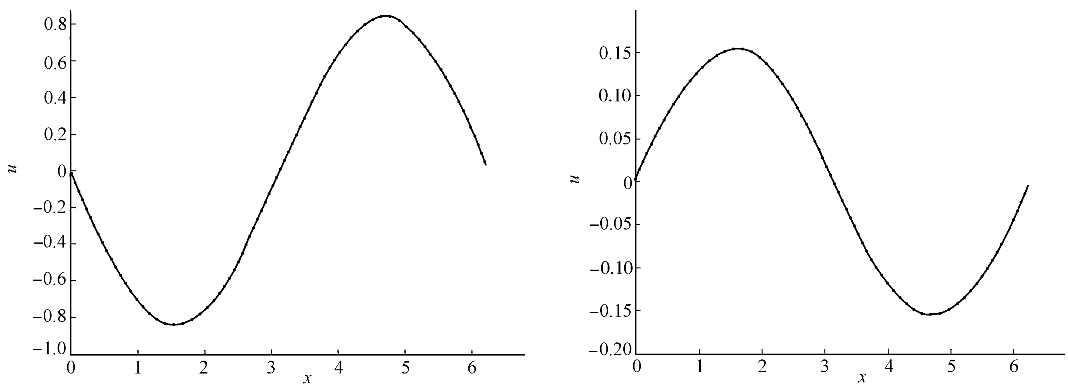
考虑初边值问题

$$\left.\begin{aligned}&\frac{\partial^2u}{\partial t^2}+\frac{\partial^4u}{\partial x^4}=0,\\&u(x,0)=\sin x,\\&u_t(x,0)=0,\\&u(x+2\pi,t)=u(x,t).\end{aligned}\right\}$$

其精确解为 $u(x,t)=\sin x\cos t$. 取 $\Delta x=\frac{2\pi}{128}$, $\Delta t=0.005$, 当 t 分别为10, 20和30时, 其精确解和由式(9)算得的数值解, 如表1所示. 从表1可知, 理论分析是正确的. 图1为时间 t 为10, 30的数值解和精确解, 图中虚线为数值解, 实线为精确解. 从图1可看出, 两者图形是重合的, 所构造的格式是有效的, 具

表 1 式(9)的数值结果比较表

t		$\frac{11\pi}{64}$	$\frac{41\pi}{64}$	$\frac{71\pi}{64}$	$\frac{101\pi}{64}$
10	精确解	- 0.431 368	- 0.758 512	0.282 675	0.813 926
	数值解	- 0.431 374	- 0.758 522	0.282 679	0.813 937
20	精确解	0.209 796	0.368 902	- 0.137 479	- 0.395 852
	数值解	0.209 815	0.368 936	- 0.137 492	- 0.395 889
30	精确解	0.079 301	0.139 442	- 0.051 966	- 0.149 629
	数值解	0.079 269	0.139 386	- 0.051 945	- 0.149 569



(a) $t=10$ (b) $t=30$

图 1 Fourier 拟谱方法的数值解与精确解

有良好的长时间数值行为及稳定性.

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Multi-Symplectic Fourier Pseudo-Spectral Method for Vibrations of Beams

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Abstract The discretization in space and in time and the mulit-symplectic consevation law are obtained for the symplectic schemes of vibration equations of beams by means of Fourier pseudo-spectral method. The conservation of the discrete local energy is proved. It is shown that the method is effective by numerical examples.

Keywords vibration of beam, multi-symplectic, Fourier pseudo-spectral, conservation law