

# 一类奇摄动非线性边值问题激波解的间接匹配

吴 钦 宽

(南京工程学院基础部, 江苏 南京 210013)

**摘要** 讨论一类非线性奇摄动方程的激波问题. 利用间接匹配法, 首先构造出外部解, 然后引入伸长变量, 最后构造出激波在区间内的激波解.

**关键词** 奇摄动, 非线性方程, 激波, 间接匹配

**中图分类号** O 241.81; O 175.14

**文献标识码** A

非线性奇摄动问题的理论和方法, 在当今国际学术界的研究中是一个十分热门的问题<sup>[1]</sup>. 许多学者做了大量的工作, 文[2~4]利用边界层理论、极值原理、重正规化法、线化法, 以及对角线技术等研究一些非线性奇摄动问题. 近年来, 对非线性奇摄动方程激波问题的研究更引起人们的广泛关注<sup>[5~10]</sup>. 本文利用间接匹配法<sup>[11]</sup>, 考虑如下—类非线性奇摄动激波问题, 有

$$\epsilon \frac{d^2 y}{dz^2} + y' \frac{dy}{dz} - y^m = 0, \quad (1)$$

边界条件  $y(-1, \epsilon) = a(\epsilon) = \sum_{i=0}^{\infty} a_i \epsilon^i$ ,  $y(1, \epsilon) = b(\epsilon) = \sum_{i=0}^{\infty} b_i \epsilon^i$ . 在式(1)中,  $\epsilon$  为正的小参数,  $s, m$  为奇数.

## 1 解的间接匹配

设问题(1)的外部解为

$$y(z, \epsilon) = y_0(z) + \epsilon y_1(z) + \dots \quad (2)$$

将式(2)代入式(1), 可得

$$\frac{dy_0}{dz} - y_0^{m-1} = 0, \quad \frac{d^2 y_0}{dz^2} + \frac{d[y_0' y_1]}{dz} - m y_0^{m-1} y_1 = 0, \quad \dots \quad (3)$$

考虑到问题(1)的边界条件, 并由式(4)可知, 问题(1)的一阶外部解的可能形式为

$$y'(z, \epsilon) = y_0'(z) + \epsilon y_1'(z), \quad y(z, \epsilon) = y_0(z) + \epsilon y_1(z). \quad (4)$$

设  $z = d(\epsilon) \in (-1, 1)$  为问题(1)的激波位置. 在  $z = d(\epsilon)$  附近引入伸长变量, 即

$$x = \epsilon^{-1} [z - d(\epsilon)]. \quad (5)$$

设问题(1)的内层解为  $w(x, \epsilon) = w_0(x) + \epsilon w_1(x) + \dots$ . 将式(5)代入问题(1), 可得

$$\frac{d^2 w}{dx^2} + w' \frac{dw}{dx} - \epsilon w^m = 0. \quad (6)$$

这时,  $w_j (j=0, 1, \dots)$  应满足方程

$$\frac{d^2 w_0}{dx^2} + w_0' \frac{dw_0}{dx} = 0, \quad \frac{d^2 w_1}{dx^2} + \frac{d[w_0' w_1]}{dx} = w_0^m, \quad \dots \quad (7)$$

为了探讨式(2), (7)的匹配, 我们定义变量的标准变换<sup>[10]</sup>为  $x = \eta^{-1} \zeta$ ,  $z = d(\epsilon) + \bar{\eta} \zeta$ ,  $\bar{\eta} \eta = \epsilon$ . 这里的  $\bar{\eta}(\epsilon)$ ,  $\eta(\epsilon)$  是标准函数<sup>[11]</sup>, 即

$$\bar{\eta}(\epsilon) = \epsilon^\alpha, \quad \eta(\epsilon) = \epsilon^{1-\alpha}, \quad 0 < \alpha < 1. \quad (8)$$

收稿日期 2005-09-13

作者简介 吴钦宽(1958-), 男, 教授, 主要从事应用数学和非线性问题的研究. E-mail: wuqk@njit.edu.cn

基金项目 国家自然科学基金资助项目(10471039)

由于我们在匹配过程中固定  $\zeta$ , 标准函数  $i$  视为新变量. 于是令  $p_i(\eta) = w_i(\eta^{-1}\zeta)$ ,  $\zeta$  为常数. 设  $p(\eta) = p_0(\eta) + \varepsilon p_1(\eta) + \dots$ , 方程(8)可写成

$$\frac{d}{d\eta}[\eta^2 \frac{dp_0}{d\eta}] - \zeta p_0 \frac{dp_0}{d\eta} = 0, \quad \frac{d}{d\eta}[\eta^2 \frac{dp_1}{d\eta}] - \zeta \frac{d}{d\eta}[p_0 p_1] = \frac{\zeta^2}{\eta^2} p_0^m, \quad \dots \quad (9)$$

为了求解方程(9), 我们设  $p_0(\eta) = \sum_{k \geq 0} A_k^0 \eta^k$ ,  $p_1(\eta) = A_{-1}^1 + \sum_{k \geq 0} A_k^1 \eta^k, \dots$ . 由  $p_i(\eta) = w_i(\eta^{-1}\zeta)$  和式(9)可得,  $p_0(\eta) = a_0(\zeta)$ ,  $p_1(\eta) = a_0^{m-1}(\zeta) \frac{\zeta}{\eta} + a_1(\zeta), \dots$ . 为了寻找匹配条件, 定义一阶外解, 有

$$\left. \begin{aligned} y'(z) &= y_0'(z) + \varepsilon y_1'(z), & y'(-1) &= a_0 + a_1 \varepsilon, & -1 \leq z < d(\varepsilon), \\ y'(z) &= y_0'(z) + \varepsilon y_1'(z), & y'(1) &= b_0 + b_1 \varepsilon, & d(\varepsilon) < z \leq 1. \end{aligned} \right\} \quad (10)$$

置  $z = d(\varepsilon) + \bar{\eta}\zeta$ , 根据 Karwowski 间接匹配法<sup>[10]</sup>, 试用一对  $y'(\eta)$ ,  $y'(\eta)$  与  $y'(z)$ ,  $y'(z)$  匹配, 有

$$\left. \begin{aligned} p'(\eta) &= a_0'(\zeta) + \varepsilon[(a_0'(\zeta))^{m-1} f_0 \frac{\zeta}{\eta} + a_1'(\zeta)] + \dots, \\ p'(z) &= a_0'(\zeta) + \varepsilon[(a_0'(\zeta))^{m-1} f_0 \frac{\zeta}{\eta} + a_1'(\zeta)] + \dots. \end{aligned} \right\} \quad (11)$$

于是, 有

$$\left. \begin{aligned} a_0'(z) &[(s+1-m)(z_0+1 + \frac{a_0^{s+1-m}}{s+1-m})]^{\frac{1}{s+1-m}}, & a_1' &= z_1, \\ a_0'(z) &[(s+1-m)(z_0+1 + \frac{b_0^{s+1-m}}{s+1-m})]^{\frac{1}{s+1-m}}, & a_1' &= z_1. \end{aligned} \right\} \quad (12)$$

在这里, 必然设  $d(\varepsilon) = z_0 + \varepsilon z_1 + \dots$ .

## 2 非线性方程的激波解

按照我们的假定, 有

$$\left. \begin{aligned} w_0(x) &= p_0'(x) + t_0'(x) = a_0' + t_0'(x), & x &\leq 0, \\ w_0(x) &= p_0'(x) + t_0'(x) = a_0' + t_0'(x), & x &\geq 0. \end{aligned} \right\} \quad (13)$$

由于函数  $t_0'(x)$ ,  $t_0'(x)$  是指数型小项<sup>[10]</sup>, 下列极限必为零, 即  $\lim_{x \rightarrow -\infty} t_0'(x) = 0$ ,  $\lim_{x \rightarrow +\infty} t_0'(x) = 0$ . 为了确定  $a_0'$ ,  $a_0'$ , 我们求解初值问题, 则

$$\frac{d^2 w_0}{dx^2} + w_0 \frac{dw_0}{dx} = 0, \quad w_0(0) = 0. \quad (14)$$

依据边界条件, 有

$$\lim_{x \rightarrow -\infty} w_0(x) = a_0', \quad \lim_{x \rightarrow +\infty} w_0(x) = a_0'. \quad (15)$$

由式(14)可得,  $\frac{dw_0}{dx} = -\frac{1}{s+1} w_0^{s+1}(0) + c_0$ . 其中,  $c_0$  为正常数. 事实上, 如果  $c_0 \leq 0$ , 则当  $x \rightarrow \pm\infty$  时, 就有  $w_0 \rightarrow \pm\infty$ . 这与式(15)矛盾. 于是, 可设  $c_0 = k^{s+1} > 0$ , 且  $k > 0$ . 则易得

$$\sqrt[2n]{2nk^{1-2n}} \cdot \frac{dw_0}{1-w_0^{2n}} = dx, \quad w_0 = \sqrt[2n]{2nk^{1-2n} w_0}, \quad (16)$$

其中  $2n = s+1$ . 从而可得

$$\begin{aligned} w_0 &= \sqrt[2n]{2nk^{1-2n}} \left\{ \frac{1}{2n} \ln \frac{\bar{w}_0 + 1}{\bar{w}_0 - 1} - \frac{1}{2n} \sum_{j=1}^{n-1} \cos \frac{2j\pi}{2n} \cdot \ln(\bar{w}_0^2 - 2cs \frac{2j\pi}{2n} \bar{w}_0 + 1) + \right. \\ &\quad \left. \frac{1}{n} \sum_{j=1}^{n-1} \sin \frac{2j\pi}{2n} \cdot \arctan \frac{\bar{w}_0 - \cos \frac{2j\pi}{2n}}{\sin \frac{2j\pi}{2n}} \right\} = x + c_1, \end{aligned} \quad (17)$$

其中,  $c_1$  由  $w_0(0)$  和式(17)确定. 最后, 由式(12), (15)和式(17)确定  $k$  和  $z_0$  的值, 要求满足  $k > 0$ ,  $|z_0| < 1$ . 为了再计算  $z_1$ ,  $w_1$ , 我们假定

$$\left. \begin{aligned} w_1(x) &= p_1'(x) + t_1'(x) = z_1 + x + t_1'(x), & x &\leq 0, \\ w_1(x) &= p_1'(x) + t_1'(x) = z_1 + x + t_1'(x), & x &\geq 0, \end{aligned} \right\}$$

$$\lim_{x \rightarrow -\infty} t_1'(x) = 0, \quad \lim_{x \rightarrow +\infty} t_1'(x) = 0, \quad (18)$$

$w_1$  是下列初值问题的解,即

$$\frac{d^2 w_1}{dx^2} + \frac{d}{dx}[w_0' w_1] = w_0^m, \quad w_0(0) = 0. \quad (19)$$

由式(19)可得

$$\left. \begin{aligned} w_1 &= \exp(-g_0(x)) \int_0^x [c_2 + g_1(s)] \exp(-g_0(s)) ds, \\ g_0(x) &= \int_0^x w_0'(s) ds, \quad g_1(x) = \int_0^x w_0^m(s) ds. \end{aligned} \right\} \quad (20)$$

由式(18),(20)和类似文[10]的推导,可得  $z_1=0$  和  $c_2=-g_1(\infty)$ . 事实上,不难得到,当  $j \geq 1$  时,  $z_j=0$ . 因此,我们得到问题(1)的一阶复合近似激波解为

$$y(z, \epsilon) = \begin{cases} y'(z) + \sum_{j=1}^1 \epsilon^j t_j'(\epsilon^{-1}[z - z_0]) + O(\epsilon^2), & -1 \leq z \leq z_0, \\ y''(z) + \sum_{j=1}^1 \epsilon^j t_j''(\epsilon^{-1}[z - z_0]) + O(\epsilon^2), & z_0 \leq z \leq 1. \end{cases} \quad (21)$$

在式(21)中,  $z_0$  是  $d(\epsilon)$  的精确位置.

继续用上述方法,我们可进一步得到问题(1)的更高阶复合近似激波解. 当  $s=1, m=1$  时, 问题(1)就是 Cole-Lagerstrom 激波问题<sup>[12]</sup>.

#### 参 考 文 献

- 1 de Jager E M, Jiang Furu. The theory of singular perturbation[M]. Amsterdam: North-Holland Publishing Co, 1996. 1~7
- 2 Glizer V Y, Fridman E. A control singularly perturbed system with small state delay[J]. J Math Anal Appl, 2000, 250(1):49~85
- 3 Kadalbajoo M K, Patdar K C. Singularly perturbed problems in partial differential equations; A survey[J]. Appl Math Comput, 2003, 134(2-3):371~429
- 4 Hamouda M. Interior layer for second-order singular equations[J]. Applicable Anal, 2002, 81(3):837~866
- 5 O'Malley R E Jr. On the asymptotic solution of the singularly perturbed boundary value problems posed by Bohé[J]. J Math Anal Appl, 2000, 242(1):18~38
- 6 Kelley W G. A singular perturbation problem of Carrier and Pearson[J]. J Math Anal Appl, 2001, (255):678~697
- 7 莫嘉琪. 一类非线性方程的激波解[J]. 数学物理学报, 2003, 23A(5):530~534
- 8 吴钦宽. 一类敏感边值问题激波位置的变化[J]. 工程数学学报, 2004, 21(4): 653~656
- 9 吴钦宽. 一类激波问题的间接匹配解[J]. 物理学报, 2005, 54(6): 2 510~2 513
- 10 Karwowski A J. Remark on indirect matching of singularly perturbed boundaryvalue problems[J]. Quarterly of Appl Math, 2003, 61(3): 401~433
- 11 Nayfeh A H. Introduction to perturbation techniques[M]. New York: John Wiley & Sons, 1981. 14~15
- 12 Kevorkian J, Cole J D. Multiple scales and singular perturbation methods[M]. New York: Springer-Verlag, 1996. 82~94

## The Indirect Matching of Shock Solution for A Class of Singularly Perturbed Nonlinear Boundary Value Problems

Wu Qinkuan

(Department of Basic Courses, Nanjing Institute of Technology, 210013, Nanjing, China)

**Abstract** In this paper, the shock problems for class of nonlinear singularly perturbed equations is considered. Using indirect matching method, we construct outer solutions, then introduce stretch variable, and at last the shock solutions in interval.

**Keywords** singular perturbation, nonlinear equation, shock, indirect matching