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IMBq 方程的多辛格式及其守恒律

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摘要 考虑非线性 IMBq 方程的多辛 Hamilton 形式,通过消去中间变量,得到新的等价于多辛 Preissman 积分的格式. 发现它具有多辛守恒律、局部能量守恒律及局部动量守恒律,最后以数值例子验证其有效性.

关键词 IMBq 方程, 多辛, 守恒律, 非线性, Hamilton 形式

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由冯康先生开创的 Hamilton 系统的辛几何算法[1]在理论上已日趋成熟,且被广泛地应用于大量的数学物理方程. 这一算法因其独特的稳定性与长期跟踪能力已解决了许多实际问题[2,3],这是传统非辛算法所不能比拟的. 因此,对辛算法的研究具有重要的理论与实际意义. 本文的目的在于用辛几何的观点,解决下列非线性 IMBq 方程的周期初值问题

$$\left.\begin{aligned} & u_{tt} - u_{xxtt} = (u^3)_{xx} + u_{xx}, \\ & u(x+80, t) = u(x, t), \\ & u(x, 0) = f(x). \end{aligned}\right\} \tag{1}$$

它用于描述一定限制条件下弹性杆的纵向形变波的传播,在离子声波方面应用较为广泛. 本文就用多辛算法研究此方程.

1 非线性 IMBq 方程的多辛形式

Reich 曾指出大量偏微分方程都具有多辛结构[4,5],从而可改写多辛 Hamilton 方程组形式为

$$\boldsymbol{M}z_t + \boldsymbol{K}z_x = \nabla_z S(z), \qquad z \in \mathbf{R}^n, \quad (x, t) \in \mathbf{R}^2. \tag{2}$$

在式(2)中,$\boldsymbol{M}, \boldsymbol{K} \in \mathbf{R}^{n\times n}$ $(n \geqslant 3)$ 是反对称矩阵,$S: \mathbf{R}^n \to \mathbf{R}$ 是光滑函数,称为 Hamilton 函数,$\nabla_z S(z)$ 为函数 $S(z)$ 的梯度. 多辛 Hamilton 系统(2)满足下列几个守恒律. (Ⅰ) 多辛守恒律. 有

$$\frac{\partial}{\partial t}\omega + \frac{\partial}{\partial x}k = 0, \tag{3}$$

其中 $\omega = \frac{1}{2}\mathrm{d}z \wedge \boldsymbol{M}\mathrm{d}z$, $k = \frac{1}{2}\mathrm{d}z \wedge \boldsymbol{K}\mathrm{d}z$. (Ⅱ) 局部能量守恒律. 有

$$\frac{\partial}{\partial t}E(x, t) + \frac{\partial}{\partial x}F(x, t) = 0, \tag{4}$$

其中 $E(x, t) = S(z) - \frac{1}{2}z^{\mathrm{T}}\boldsymbol{K}z_x$, $F(x, t) = \frac{1}{2}z^{\mathrm{T}}\boldsymbol{K}z_t$. (Ⅲ) 局部动量守恒律. 有

$$\frac{\partial}{\partial t}I(x, t) + \frac{\partial}{\partial x}G(x, t) = 0, \tag{5}$$

其中 $I(x, t) = \frac{1}{2}z^{\mathrm{T}}\boldsymbol{M}z_x$, $G(x, t) = S(z) - \frac{1}{2}z^{\mathrm{T}}\boldsymbol{M}z_t$. 引入正则动量 $w_x = v$, $u_t = v_x$, $u_x = q$, $v_t = p$, 则可得方程组(1)的多辛 Hamilton 方程组

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$$\left.\begin{aligned} w_t - p_x = V'(u) + u, \qquad - q_t - w_x = - v_{xx} - v,\\ - u_t + v_x = 0, \qquad u_x = q, \qquad v_t = p. \end{aligned}\right\} \tag{6}$$

其中 $V(u) = u^4/4$. 令状态变量 $z = (u, v, w, p, q)^{\mathrm{T}}$, 于是 Hamilton 方程组(6) 可写成形式(2). 此时

$$S = \frac{1}{2}(u^2 - v^2 + v_x^2) + pq + V(u),$$

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2 几个局部守恒律

易证方程组(6) 满足上述几个局部守恒律, 其具体形式有 3 种(因篇幅关系证明从略). (i) 相应于式(3) 的多辛守恒律为

$$\frac{\partial}{\partial t}(\mathrm{d}u \wedge \mathrm{d}w + \mathrm{d}q \wedge \mathrm{d}v) + \frac{\partial}{\partial x}(\mathrm{d}p \wedge \mathrm{d}u + \mathrm{d}w \wedge \mathrm{d}v) = 0. \tag{7}$$

(ii) 相应于式(4) 的局部能量守恒律为

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(u^2 + v_x^2 - v^2) + V(u) - wv_x\right] + \frac{\partial}{\partial x}(2pu_t + wv_t) = 0. \tag{8}$$

(iii) 相应于式(5) 的局部动量守恒律为

$$\frac{\partial}{\partial t}(uw_x + qv_x) + \frac{\partial}{\partial x}\left[\frac{1}{2}(u^2 - v^2 + u_t^2) + V(u) - uw_t\right] = 0. \tag{9}$$

3 Preissman 多辛积分及其等价格式

引入正则动量 $\frac{1}{\Delta x}(w_{i+1}^{j+1/2} - w_i^{j+1/2}) = v_{i+1/2}^{j+1/2}$, $\frac{1}{\Delta t}(u_{i+1/2}^{j+1} - u_{i+1/2}^{j}) = \frac{1}{\Delta x}(v_{i+1}^{j+1/2} - v_i^{j+1/2})$, $p_{i+1/2}^{j+1/2} = \frac{1}{\Delta t} \cdot (v_{i+1/2}^{j+1} - v_{i+1/2}^{j})$, $q_{i+1/2}^{j+1/2} = \frac{1}{\Delta x}(u_{i+1}^{j+1/2} - u_i^{j+1/2})$, 则得到方程组(6) 的中心 Preissman 格式为

$$\left.\begin{aligned} &\frac{1}{\Delta t}(w_{i+1/2}^{j+1} - w_{i+1/2}^{j}) - \frac{1}{\Delta x}(p_{i+1}^{j+1/2} - p_i^{j+1/2}) = V'(u_{i+1/2}^{j+1/2}) + u_{i+1/2}^{j+1/2},\\ &\frac{1}{\Delta t}(q_{i+1/2}^{j+1} - q_{i+1/2}^{j}) + \frac{1}{\Delta x}(w_{i+1}^{j+1/2} - w_i^{j+1/2}) = (v_{i+1/2}^{j+1/2})_{xx} + v_{i+1/2}^{j+1/2},\\ &\frac{1}{\Delta t}(u_{i+1/2}^{j+1} - u_{i+1/2}^{j}) = \frac{1}{\Delta x}(v_{i+1}^{j+1/2} - v_i^{j+1/2}),\\ &\frac{1}{\Delta t}(v_{i+1/2}^{j+1} - v_{i+1/2}^{j}) = p_{i+1/2}^{j+1/2}, \qquad \frac{1}{\Delta x}(u_{i+1}^{j+1/2} - u_i^{j+1/2}) = q_{i+1/2}^{j+1/2}. \end{aligned}\right\} \tag{10}$$

消去中间变量 v, w, p, q, 就得到一个与 Preissman 多辛积分等价的新格式为

$$\begin{aligned} &\frac{\delta_t^2(u_{i+3}^{j+1} + u_{i+3}^{j}) + 4\delta_t^2(u_{i+2}^{j+1} + u_{i+2}^{j}) + 6\delta_t^2(u_{i+1}^{j+1} + u_{i+1}^{j}) + 4\delta_t^2(u_i^{j+1} + u_i^{j}) + \delta_t^2(u_{i-1}^{j+1} + u_{i-1}^{j})}{32\Delta t^2} - \\ &\frac{\delta_t^2\delta_x^2(u_{i+3}^{j+1} + u_{i+3}^{j}) - 4\delta_t^2\delta_x^2(u_{i+2}^{j+1} + u_{i+2}^{j}) - 10\delta_t^2\delta_x^2(u_{i+1}^{j+1} + u_{i+1}^{j}) - 4\delta_t^2\delta_x^2(u_i^{j+1} + u_i^{j}) + \delta_t^2\delta_x^2(u_{i-1}^{j+1} + u_{i-1}^{j})}{32\Delta t^2\Delta x^2} = \\ &\frac{1}{8}\left[V'(u_{i+1}^{j+1}) + V'(u_i^{j+1}) + 2(V'(u_{i+1}^{j}) + V'(u_i^{j})) + V'(u_{i+1}^{j-1}) + V'(u_i^{j-1})\right]_{xx} + \\ &\frac{1}{8}\left[(u_{i+1}^{j+1} + u_i^{j+1}) + 2(u_{i+1}^{j} + u_i^{j}) + u_{i+1}^{j-1} + u_i^{j-1}\right]_{xx}. \end{aligned} \tag{11}$$

初始条件的离散化处理从略. 其中 $u_i^j \approx u(i\Delta x, j\Delta t)$. Δx 和 Δt 分别是空间步长和时间步长, 且 $u_{i+1/2}^{j} = \frac{1}{2} \cdot (u_i^j + u_{i+1}^j)$, $u_i^j = u_{i+1/2}^{j+1/2} = \frac{1}{2}(u_{i+1/2}^{j+1} + u_{i+1/2}^{j}) = \frac{1}{4}(u_i^j + u_i^{j+1} + u_{i+1}^{j} + u_{i+1}^{j+1})$, $u_i^{j+1/2} = \frac{1}{2}(u_i^{j+1} + u_i^j)$,

$(u_i^j)_{xx} = \frac{\delta_x^2 u_i^j}{\Delta x^2}$, $\delta_x^2 u_i^j = u_{i+1}^j - 2u_i^j + u_{i-1}^j$, $\delta_t^2 u_i^j = u_i^{j+1} - 2u_i^j + u_i^{j-1}$ 等等.

注　格式(10)具离散多辛守恒律[6]. 有

$$\frac{(du_{i+1/2}^{j+1}\wedge dw_{i+1/2}^{j+1} - du_{i+1/2}^{j}\wedge dw_{i+1/2}^{j}) + (dq_{i+1/2}^{j+1}\wedge dv_{i+1/2}^{j+1} - dq_{i+1/2}^{j}\wedge dv_{i+1/2}^{j})}{\Delta t} + \frac{(dp_{i+1}^{j+1/2}\wedge du_{i+1}^{j+1/2} - dp_{i}^{j+1/2}\wedge du_{i}^{j+1/2}) + (dw_{i+1}^{j+1/2}\wedge dv_{i+1}^{j+1/2} - dw_{i}^{j+1/2}\wedge dv_{i}^{j+1/2})}{\Delta x} = 0. \tag{12}$$

4　数值例子

在非线性 IMBq 方程的周期初值问题(1)中取 $f(x) = A\,\mathrm{sech}kx$, 设 $u(-40, t) = u(40, t)$, 其精确解为 $u(x, t) = A\,\mathrm{sech}(kx + ct)$, 其中 A, k, c 满足关系式 $A^2 = 2c^2$, $k^2 = \frac{c^2}{c^2+1}$. 取 $\Delta x = 0.2$, $\Delta t = 0.001$, $A = \sqrt{2}$, $k = 1/\sqrt{2}$, 用格式(11)进行数值模拟. 为方便起见, 除了 $t = 0$ 时的初始层采用精确值外, 第 2, 3 层也用精确值进行计算. 图 1 画出了计算 10 000 层后格式(11)在区间 $[-40, 40]$ 上的数值解与原方程的精确解的结果比较情况. 结果表明多辛格式(11)是有效的, 具有良好的长时间数值行为, 数值结果与理论分析相符.

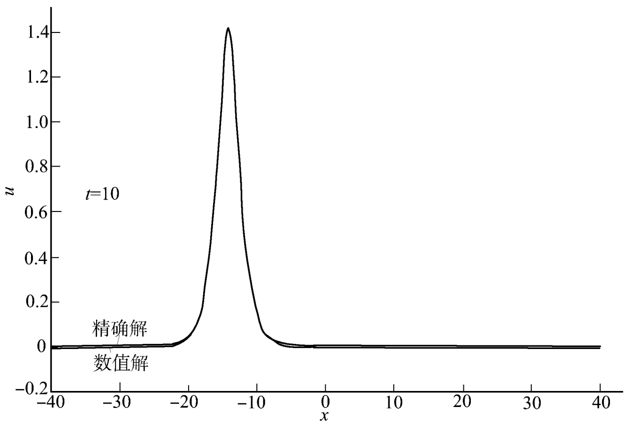

图 1　数值结果比较图

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Multi-Symplectic Scheme for IMBq Equation and the Conservation Laws to Which It Subjects

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Abstract　In consideration of multi-symplectic Hamiltonian formulation for the nonlinear IMBq equation, by eliminating intermediate variables, the author gets a new scheme which is equivalent to the multi-symplectic Preissman integrator. Scheme subjects to conservation laws of multic-symplecticity, local energy and local momentum. Finally, its effectiveness is verified by numerical example.

Keywords　IMBq equation, multi-symplecticity, conservation law, nonlinear, Hamiltonian formulation