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解四阶抛物型方程的高精度差分格式

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摘要 对四阶抛物型方程构造一族含参数高精度三层差分格式. 当参数满足一定的条件时, 差分格式稳定. 局部截断误差阶数最高可达 $O(\tau^2 + h^6)$. 最后, 用数值例子说明对稳定性所作的分析是正确的.

关键词 四阶抛物型方程, 差分格式, 绝对稳定.

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考虑下列四阶抛物型方程初边值问题

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} &= 0, \quad 0 < x < 1, \quad 0 < t \leq T, \\ u(x, 0) &= f(x), \quad 0 \leq x \leq 1, \\ u(0, t) = \frac{\partial^2 u(0, t)}{\partial x^2} = u(1, t) = \frac{\partial^2 u(1, t)}{\partial x^2} &= 0, \quad 0 \leq t \leq T. \end{aligned} \right\} \tag{1}$$

文[1~ 4]等对求解四阶抛物形方程的差分格式进行了研究, 得到了不同的结果, 其中有的截断误差阶达到了 $O(\tau^2 + h^6)$. 本文构造了一族新的三层含参数隐式格式, 对文[4]的结果进行了改进. 当诸参数满足定理的条件时, 所给出的格式绝对稳定.

1 差分格式的构造

以下分别用 τ, h 表示时间 t 及空间方向 x 的步长, 用 u_j^n 表示 $u(jh, n\tau)$ 的差分逼近. 网域由点集 $(x_j, t_n)(j = 0, 1, 2, \dots, J; n = 0, 1, 2, \dots)$ 所组成, 其中

$$x_j = jh, \quad t_n = n\tau, \quad h = 1/J.$$

又设 $r = \tau/h^4$ 为网格比, 初边界条件的离散化处理同文[1]

用如下含参数并具有对称形式的差分方程逼近微分方程(1), 即

$$\begin{aligned} &\theta_{12} \frac{u_{j-2}^{n+1} - u_{j-2}^{n-1}}{\tau} + \theta_{11} \frac{u_{j-1}^{n+1} - u_{j-1}^{n-1}}{\tau} + \theta_{10} \frac{u_j^{n+1} - u_j^{n-1}}{\tau} + \\ &\theta_{11} \frac{u_{j+1}^{n+1} - u_{j+1}^{n-1}}{\tau} + \theta_{12} \frac{u_{j+2}^{n+1} - u_{j+2}^{n-1}}{\tau} + \theta_{22} \frac{u_{j-2}^n - u_{j-2}^{n-1}}{\tau} + \\ &\theta_{21} \frac{u_{j-1}^n - u_{j-1}^{n-1}}{\tau} + \theta_{20} \frac{u_j^n - u_j^{n-1}}{\tau} + \theta_{21} \frac{u_{j+1}^n - u_{j+1}^{n-1}}{\tau} + \theta_{22} \frac{u_{j+2}^n - u_{j+2}^{n-1}}{\tau} + \\ &\theta_3 \frac{\delta_x^4 u_j^{n+1}}{h^4} + \theta_4 \frac{\delta_x^4 u_j^n}{h^4} + \theta_5 \frac{\delta_x^4 u_j^{n-1}}{h^4} = 0, \end{aligned} \tag{2}$$

其中 $\theta_{10}, \theta_{11}, \theta_{12}, \theta_{20}, \theta_{21}, \theta_{22}, \theta_3, \theta_4, \theta_5$ 为待定参数. 适当选取这些参数, 可以使差分格式(2)逼近微分方程(1), 具有尽可能高阶的离散误差, 而且有很好的稳定性. δ_x^4 表示 x 方向的四阶中心差分算子, 即

$$\delta_x^4 u_j^n = u_{j-2}^n - 4u_{j-1}^n + 6u_j^n - 4u_{j+1}^n + u_{j+2}^n.$$

当微分方程(1)的解充分光滑时,有如下的关系式成立,即

$$\frac{\partial^{p+q} u}{\partial x^p \partial t^q} = (-1)^q \frac{\partial^{p+4q} u}{\partial x^{p+4q}}, \quad p, q = 0, 1, 2, \dots \quad (3)$$

将格式(2)中各节点上的 u 在网点 (x_j, t_n) 或 $(jh, n\tau)$ 处展开的 Taylor 级数代入,经整理后得

$$\begin{aligned} & (\theta_{10} + 2\theta_{11} + 2\theta_{12} + \theta_{20} + 2\theta_{21} + 2\theta_{22}) \frac{\partial u}{\partial t} + (\theta_3 + \theta_4 + \theta_5) \frac{\partial^4 u}{\partial x^4} + \\ & \tau [-\frac{1}{2}(\theta_{20} + 2\theta_{21} + 2\theta_{22}) \frac{\partial^2 u}{\partial t^2} + (\theta_3 - \theta_5) \frac{\partial^5 u}{\partial x^4 \partial t}] + \\ & \tau^2 [\frac{1}{3!}(\theta_{10} + 2\theta_{11} + 2\theta_{12} + \theta_{20} + 2\theta_{21} + 2\theta_{22}) \frac{\partial^3 u}{\partial t^3} + \frac{1}{2}(\theta_3 + \theta_5) \frac{\partial^6 u}{\partial x^4 \partial t^2}] + \\ & h^2 [\frac{3}{3!}(2\theta_{11} + 8\theta_{12} + 2\theta_{21} + 8\theta_{22}) \frac{\partial^3 u}{\partial x^2 \partial t} + \frac{1}{6}(\theta_3 + \theta_4 + \theta_5) \frac{\partial^6 u}{\partial x^6}] + \\ & \tau h^2 [-\frac{6}{4!}(2\theta_{21} + 8\theta_{22}) \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{1}{6}(\theta_3 - \theta_5) \frac{\partial^7 u}{\partial x^6 \partial t}] + \\ & h^4 [\frac{5}{5!}(2\theta_{11} + 2^5 \theta_{12} + 2\theta_{21} + 2^5 \theta_{22}) \frac{\partial^5 u}{\partial x^4 \partial t} + \frac{1}{80}(\theta_3 + \theta_4 + \theta_5) \frac{\partial^8 u}{\partial x^8}] + \tau h^3 \cdot 0 + \\ & h^6 [\frac{7}{7!}(2\theta_{11} + 2\theta_{12} + 2\theta_{21} + 2\theta_{22}) \frac{\partial^7 u}{\partial x^6 \partial t} + \frac{17}{16 \times 1890}(\theta_3 + \theta_4 + \theta_5) \frac{\partial^{10} u}{\partial x^{10}}] + \\ & O(\tau h^4 + \tau^3 + h^7) = 0. \end{aligned} \quad (4)$$

由关系式(3)及 $\tau = O(h^4)$, 当下列诸条件同时成立时, 差分格式(2)的截断误差为 $O(\tau^2 + h^6)$. 有

$$\left. \begin{aligned} \theta_{10} + 2\theta_{11} + 2\theta_{12} + \theta_{20} + 2\theta_{21} + 2\theta_{22} &= 1, \\ \theta_3 + \theta_4 + \theta_5 &= 1, \\ \theta_{20} + 2\theta_{21} + 2\theta_{22} &= -2(\theta_3 - \theta_5), \\ 6(\theta_{11} + 4\theta_{12} + \theta_{21} + 4\theta_{22}) &= 1, \\ 20(\theta_{11} + 16\theta_{12} + \theta_{21} + 16\theta_{22}) &= 3. \end{aligned} \right\} \quad (5)$$

解方程组(5), 得

$$\left. \begin{aligned} \theta_{10} &= \frac{79}{120} - \beta_1, & \beta_1 &= \theta_{20}, \\ \theta_{11} &= \frac{31}{180} - \beta_2, & \beta_2 &= \theta_{21}, \\ \theta_{12} &= \frac{-1}{720} - \beta_3, & \beta_3 &= \theta_{22}, \\ \theta_3 &= \frac{1}{2}(1 - \beta_4 - \beta), & \beta_4 &= \theta_4, \\ \theta_5 &= \frac{1}{2}(1 - \beta_4 + \beta), & \beta &= \frac{1}{2}\beta_1 + \beta_2 + \beta_3. \end{aligned} \right\} \quad (6)$$

将式(6)代入差分格式(2), 可得如下含参数 $(\beta_1, \beta_2, \beta_3, \beta_4)$ 的三层隐格式, 即

$$\begin{aligned} & (\frac{1}{720} + \beta_3)(u_{j-2}^{n+1} + u_{j+2}^{n+1}) - (\frac{31}{180} - \beta_2)(u_{j-1}^{n+1} + u_{j+1}^{n+1}) - \\ & (\frac{79}{120} - \beta_1)u_j^{n+1} - (1 - \beta_4 - \beta)r\delta_x^4 u_j^{n+1} = \\ & 2\beta_3(u_{j-2}^n + u_{j+2}^n) + 2\beta_2(u_{j-1}^n + u_{j+1}^n) + 2\beta_1(u_j^n + 2\beta_4 r\delta_x^4 u_j^n + \\ & (\frac{1}{720} - \beta_3)(u_{j-2}^{n-1} + u_{j+2}^{n-1}) - (\frac{31}{180} + \beta_2)(u_{j-1}^{n-1} + u_{j+1}^{n-1}) - \\ & (\frac{79}{120} + \beta_1)u_j^{n-1} + (1 - \beta_4 + \beta)r\delta_x^4 u_j^{n-1}, \end{aligned} \quad (7)$$

其截断误差为 $O(\tau^2 + h^6)$.

当参数 $(\beta_1, \beta_2, \beta_3, \beta_4)$ 取不同的值时, 可得不同得差分格式.

(I) 当参数 $\beta_1 = \frac{79}{120}$, $\beta_2 = \frac{31}{180}$, $\beta_3 = -\frac{1}{720}$ 以及 $\beta_4 = \frac{1}{2}$ 时, 式(7)为文[2]的两层 10 点格式. 它们

具有同样的截断误差.

(II) 类似地, 还可以得到其它不同的差分格式. 本文从略.

2 差分格式的稳定性

为了证明格式的稳定性需要引用以下引理.

引理(Miller 准则) 实系数二次方程为

$Ax^2 + Bx + C = 0, \quad A > 0$

的两个根按模小于等于 1 的充要条件是 $A - C \geq 0, A + B + C \geq 0, A - B + C \geq 0$.

定理 1 格式(7) 绝对稳定的一个充分条件为

$$\left. \begin{aligned} \beta_1 &\leq 2(\beta_2 + \beta_3), \\ \beta_2 &\geq 4\beta_3, \\ \beta_3 &\leq 0, \\ \beta_4 &\leq \frac{1}{2}. \end{aligned} \right\} \tag{8}$$

证明 由 Fourier 分析法^[5]可知格式(7)的特征方程为

$A\lambda^2 + B\lambda + C = 0, \tag{9}$

其中

$$\left. \begin{aligned} A &= -\left(\frac{1}{720} + \beta_3\right)(1 - 8s^2 + 8s^4) + \left(\frac{31}{180} - \beta_2\right)(1 - 2s^2) + \\ &\quad \frac{1}{2}\left(\frac{79}{120} - \beta_1\right) + 8(1 - \beta_4 - \beta)rs^4, \\ B &= 2\beta_3(1 - 8s^2 + 8s^4) + 2\beta_2(1 - 2s^2) + \beta_1 + 16\beta_4rs^4, \\ C &= \left(\frac{1}{720} - \beta_3\right)(1 - 8s^2 + 8s^4) - \left(\frac{31}{180} + \beta_2\right)(1 - 2s^2) - \\ &\quad \frac{1}{2}\left(\frac{79}{120} + \beta_1\right) + 8(1 - \beta_4 + \beta)rs^4, \\ A + B + C &= 16rs^4 \geq 0. \end{aligned} \right\} \tag{10}$$

$A - C = 1 - \frac{2}{3}s^2 - \frac{1}{45}s^4 - 16\beta rs^4 \geq 0, \tag{11}$

当定理的条件(8)满足时, 有

$$\begin{aligned} \beta &= \frac{1}{2}\beta_1 + \beta_2 + \beta_3 \leq \\ &\quad \frac{1}{2}[-2(\beta_2 + \beta_3)] + \beta_2 + \beta_3 = 0, \end{aligned} \tag{12}$$

也即 $\beta \leq 0$. 同时有

$$\begin{aligned} A - C &= 1 - \frac{2}{3}s^2 - \frac{1}{45}s^4 - 16\beta rs^4 \geq \\ &\quad 1 - \frac{2}{3} - \frac{1}{45} - 16\beta rs^4 = \frac{14}{45} - 16\beta rs^4 > 0, \end{aligned} \tag{13}$$

$$A - B + C = 16[(1 - 2\beta_4)r - 2\beta_3]s^4 + 8(\beta_2 + 4\beta_3)s^2 - 4\beta \geq 0. \tag{14}$$

由引理知, 特征方程(9)的特征根的模小于等于 1. 又因 $A - C > 0$, 因此特征方程(9)无重根. 即对于任意的 $r > 0$, 格式(7)绝对稳定. 定理得证.

3 数值例子

考虑下列四阶抛物型初边值问题

$$\left. \begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} &= 0, \quad 0 < x < \pi, \quad t > 0, \\ u(x, 0) &= \sin x, \quad 0 \leq x \leq \pi, \\ u(0, t) &= \frac{\partial^2 u(0, t)}{\partial x^2} = u(\pi, t) = \frac{\partial^2 u(\pi, t)}{\partial x^2} = 0, \quad t > 0. \end{aligned} \right\} \tag{15}$$

其精确解为

$$u(x,t)=e^{-t}\sin x.$$

(16)

初始值以及边界条件处理同文〔2〕.

取 $h=\pi/32$, 而时间步长按 $\tau=r\cdot h^4$ 进行计算. 因本文所构造的格式为三层格式, 除初始层网格函数值为已知外, 还需要用其它方法预先计算出第 1 层网格上的函数值. 为简化计算, 第 1 层网格函数值按精确值进行计算.

下面给出精度比较表. 其值等于精确解减去差分格式解的数值. 计算结果, 如表 1 所示.

表 1 精度比较表($n=1\,000$)

r	β_1	β_2	β_3	β_4	x			
					$5\pi/32$	$13\pi/32$	$21\pi/32$	$29\pi/32$
1	0	0	0	1/2	1.697×10^{-11}	3.450×10^{-11}	3.156×10^{-11}	1.041×10^{-11}
1	-9	5	-1	1/2	2.103×10^{-8}	4.269×10^{-8}	3.934×10^{-8}	1.295×10^{-8}
1/16	-9	5	-1	1/2	8.893×10^{-10}	1.805×10^{-10}	1.663×10^{-10}	5.470×10^{-10}
1/16	-9	5	-1	5	8.856×10^{-11}	1.799×10^{-10}	1.655×10^{-10}	5.439×10^{-11}

由表 1 可以看出, 由定理 1 的稳定性条件, 仅仅是格式(7)的一个充分条件. 不同的参数对格式稳定性的影响不同.

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A Family of High Accurate Difference Schemes for Solving

Four-Order Parabolic Equation

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Abstract A family of high accurate and three layer difference schemes containing parameters are constructed for solving four order parabolic equation. These difference schemes are stable when the parameters satisfy a certain condition. The local truncation error can reach the order of $O(\tau^2+h^6)$ as the maximum. The analysis of stability is correct, as illustrated by numerical example.

Keywords four order parabolic equation, difference scheme, absolutely stable