

# 具松弛项的欧拉方程组的整体光滑解

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**摘要** 研究具松弛项可压缩的欧拉方程组柯西问题. 在关于压力函数和次特征条件的假设下, 如果初值的  $C^1$  模具小性, 且初始密度离开真空状态, 证明其柯西问题存在唯一的整体光滑解.

**关键词** 气动力学方程组, 松弛项, 整体光滑解

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## 1 问题的提出

考虑具松弛项的可压缩的欧拉方程组的柯西问题

$$\left. \begin{aligned} \rho_t + (\rho u)_x &= 0, \quad (x, t) \in \mathbf{R} \times \mathbf{R}^+, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x &= \frac{1}{\varepsilon}(\theta(\rho) - \rho u). \end{aligned} \right\} \quad (1)$$

$$(\rho(x, 0), u(x, 0)) = (\rho_0(x), u_0(x)), \quad x \in \mathbf{R} \quad (2)$$

上式中,  $\rho$ ,  $u$  和  $p(\rho)$  分别表示气体的密度、速度和压力,  $\varepsilon > 0$  为常数. 假设  $p(\rho), f(\rho)$  满足

$$\left. \begin{aligned} p(\rho) \in C^3(\mathbf{R}^+), \quad p'(\rho) > 0, \quad p''(\rho) > 0, \quad 2p'(\rho)p'''(\rho) \leq [p''(\rho)]^2, \\ \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau < +\infty, \quad \int_\rho^\infty \frac{\sqrt{p'(\tau)}}{\tau} d\tau = +\infty. \end{aligned} \right\} \quad (P)$$

$$-\sqrt{p'(\rho)} < \theta'(\rho) < \sqrt{p'(\rho)}. \quad (SC)$$

又假设初值满足如下条件

$$\left. \begin{aligned} \rho_0(x), u_0(x) \in C^1(\mathbf{R}), \quad 0 < \rho_* \leq \rho_0(x) \leq \rho^*, \quad |u_0(x)| \leq U_0, \\ |p'(\rho_0(x))| \leq \frac{\rho_1}{\varepsilon}, \quad |u_0'(x)| \leq \frac{U_1}{\varepsilon}, \end{aligned} \right\} \quad (I)$$

其中  $\rho_*$ ,  $\rho^*$ ,  $U_0$ ,  $\rho_1$ ,  $U_1$  为与  $\varepsilon$  无关的正常数.

方程组(1)的特征值为

$$\lambda = u - \sqrt{p'(\rho)}, \quad \mu = u + \sqrt{p'(\rho)}. \quad (3)$$

因此, 在非真空状态下, 它是严格双曲型方程组, Riemann 不变量

$$r = u - \Phi(\rho), \quad s = u + \Phi(\rho), \quad \Phi(\rho) = \int_0^\rho \frac{\sqrt{p'(\tau)}}{\tau} d\tau. \quad (4)$$

于是, 在光滑解意义下, 柯西问题(1), (2)可化为

$$\left. \begin{aligned} r_t + \lambda r_x &= -\frac{1}{2\varepsilon}(r + s - F(s - r)), \\ s_t + \mu s_x &= -\frac{1}{2\varepsilon}(r + s - F(s - r)), \end{aligned} \right\} \quad (5)$$

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$$(r_0(x, 0), s_0(x, 0)) = (r_0(x), s_0(x)). \quad (6)$$

在式(5)中  $F(\theta) = 2f(\Phi^{-1}(\frac{\theta}{2}))$ ,  $\Phi^{-1}(\theta)$  为函数  $\Phi(\cdot)$  的反函数.

$$r_0(x) = u_0(x) - \Phi(\rho_0(x)), \quad s_0(x) = u_0(x) + \Phi(\rho_0(x)). \quad (7)$$

如果光滑解离开真空状态, 则柯西问题(1), (2)与柯西问题(5), (6)是等价的. 具松驰项的一阶双曲型方程组的柯西问题的整体光滑解, 已有不少研究结果<sup>[1~5]</sup>. 值得指出, 文[1]对于拉格朗日坐标具松驰项的气动力学方程组柯西问题, 得到整体光滑解存在性的结果. 本文将在欧拉坐标下, 研究其整体光滑解的存在性.

## 2 解的先验估计

在先验假设

$$0 < \rho(x, t) < +\infty \quad (H)$$

成立下, 由文[6]可得到仅依赖于初值的  $C^1$ -模的正数  $t_1 > 0$ , 使在区域

$$\pi(t_1) = \{(x, t) : x \in \mathbf{R}, \quad 0 \leq t \leq t_1\}$$

上, 柯西问题(5), (6)存在唯一的光滑解  $(r(x, t), s(x, t))$ .

引理 1 在假设条件(P), (SC)和先验假设(H)下, 如果

$$|r_0(x)| \leq M_0, \quad |s_0(x)| \leq M_0, \quad (8)$$

其中  $M_0$  为与  $\varepsilon$  无关的正常数. 那么, 则柯西问题(5), (6)的光滑解在其存在区域上有估计式

$$|r(x, t)| \leq M, \quad |s(x, t)| \leq M. \quad (9)$$

其中,  $M = M_0 + \frac{1}{2} \sup_{|t| \leq 2M_0} |F(\theta)|$ .

证明 令

$$\bar{r} = re^{\frac{t}{\varepsilon}}, \quad \bar{s} = se^{\frac{t}{\varepsilon}}. \quad (10)$$

则由式(5)得

$$\left. \begin{aligned} \bar{r}_t + \bar{\kappa}_x &= -\frac{1}{2\varepsilon} [(\bar{s} - \bar{r}) - e^{\frac{t}{\varepsilon}} F(e^{-\frac{t}{\varepsilon}}(\bar{s} - \bar{r}))], \\ \bar{s}_t + \bar{\mu}_x &= \frac{1}{2\varepsilon} [(\bar{s} - \bar{r}) + e^{\frac{t}{\varepsilon}} F(e^{-\frac{t}{\varepsilon}}(\bar{s} - \bar{r}))]. \end{aligned} \right\} \quad (11)$$

其初值

$$(\bar{r}_0(x, 0), \bar{s}_0(x, 0)) = (r_0(x), s_0(x)). \quad (12)$$

设  $x = x_\lambda(\alpha, \tau)$  和  $x = x_\mu(\beta, \tau)$ , 分别表示过点  $(\alpha, 0)$  和  $(\beta, 0)$  的  $\lambda$  特征曲线和  $\mu$  特征曲线, 并相交于点  $(x, t)$ . 沿  $\lambda$  特征曲线和  $\mu$  特征曲线, 从  $\tau = 0$  至  $\tau = t$ , 积分式(11)得到

$$\begin{aligned} \bar{s}(x, t) - \bar{r}(x, t) &= s_0(\beta) - r_0(\alpha) + \frac{1}{2\varepsilon} \int_0^t [G_+(\bar{s} - \bar{r}, \tau)(x_\mu(\beta, \tau), \tau) - \\ &\quad G_-(\bar{s} - \bar{r}, \tau)(x_\lambda(\alpha, \tau), \tau)] d\tau. \end{aligned} \quad (13)$$

其中

$$G_\pm(\theta, \tau) = \theta \pm e^{\frac{\tau}{\varepsilon}} F(e^{-\frac{\tau}{\varepsilon}} \theta). \quad (14)$$

由次特征条件(SC)得

$$\frac{\partial G_\pm(\theta, \tau)}{\partial \theta} = 1 \pm \frac{\theta'(\rho)}{\sqrt{p'(\rho)}} > 0. \quad (15)$$

令

$$R(\tau) = \sup_{x \in \mathbf{R}} |\bar{s}(x, \tau) - \bar{r}(x, \tau)|, \quad (16)$$

于是由式(8), (13), (15), (16)得

$$R(t) \leq 2M_0 + \frac{1}{\varepsilon} \int_0^t R(\tau) d\tau. \quad (17)$$

利用 Gronwall 不等式, 由式(17)得

$$R(t) \leq 2M_0 e^{\frac{t}{\varepsilon}}. \tag{18}$$

则

$$|\bar{s}(x,t) - \bar{r}(x,t)| \leq 2M_0 e^{\frac{t}{\varepsilon}}. \tag{19}$$

另一方面, 由式(8), (11), (14), (19)得

$$|\bar{r}(x,t)| \leq M_0 + \frac{1}{\varepsilon} \int_0^t M e^{\frac{\tau}{\varepsilon}} d\tau \leq M e^{\frac{t}{\varepsilon}}. \tag{20}$$

类似有

$$|\bar{s}(x,t)| \leq M e^{\frac{t}{\varepsilon}}. \tag{21}$$

再由式(10), (19), (20), (21), 有式(9)成立. 引理 1 证毕. 引用文[7]的引理 1, 得

引理 2 如果  $p(\rho)$  满足条件(P), 则有

$$\rho \geq \rho_* \exp\{-\rho_*^{\frac{1}{2}}(p'_*)^{-\frac{1}{4}}g(\rho)\}, \quad \forall 0 < \rho < \rho_*, \quad p'_* = p'(\rho_*), \tag{22}$$

$$g(\rho) = \int_{\rho}^{\rho_*} \frac{[p'(\tau)]^{-\frac{1}{4}}}{\tau^{\frac{3}{2}}} d\tau. \tag{23}$$

引理 3 在引理 1 的假设条件下, 如果初值  $(\rho_0(x), u_0(x))$  满足条件(I), 且

$$r'_0(x) \leq \frac{M_1}{\varepsilon}, \quad s'_0(x) \leq \frac{M_1}{\varepsilon}. \tag{24}$$

则柯西问题(1), (2)的光滑解  $(\rho(x,t), u(x,t))$ , 使得

$$g(\rho(x,t)) = \int_{\rho(x,t)}^{\rho_*} \frac{[p'(\tau)]^{-\frac{1}{4}}}{\tau^{\frac{3}{2}}} d\tau \leq M_2, \tag{25}$$

其中,  $M_2 = \frac{[p'(\rho_*)]^{-\frac{1}{4}}}{\rho_*^{\frac{1}{2}}} M_1.$

证明 由式(5)两边对  $x$  求偏导数得

$$\left. \begin{aligned} D_N x &= - \left( \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right) r_x^2 + \left( \frac{\rho''(\rho)}{4p'(\rho)} - \frac{1}{2} \right) r_x s_x - \frac{1}{2\varepsilon} \left[ r_x + s_x - \frac{\vartheta'(\rho)}{\sqrt{p'(\rho)}}(s_x - r_x) \right], \\ D_M x &= - \left( \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right) s_x^2 + \left( \frac{\rho''(\rho)}{4p'(\rho)} - \frac{1}{2} \right) r_x s_x - \frac{1}{2\varepsilon} \left[ r_x + s_x - \frac{\vartheta'(\rho)}{\sqrt{p'(\rho)}}(s_x - r_x) \right]. \end{aligned} \right\} \tag{26}$$

其中,  $D_\lambda = \frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}, D_\mu = \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x}$ . 又由式(4), (5) 得

$$D_\lambda \rho = -\rho_x, \quad D_\mu \rho = -\rho r_x. \tag{27}$$

因此, 由式(23), (26)和(27)我们有

$$\left. \begin{aligned} D_\lambda W &= - \left( \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right) \frac{[p'(\rho)]^{-\frac{1}{4}}}{\rho^{\frac{1}{2}}} r_x^2 + \frac{1}{2\varepsilon} \left( 1 - \frac{\vartheta'(\rho)}{\sqrt{p'(\rho)}} \right) (Z - W), \\ D_\mu Z &= - \left( \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right) \frac{[p'(\rho)]^{-\frac{1}{4}}}{\rho^{\frac{1}{2}}} s_x^2 + \frac{1}{2\varepsilon} \left( 1 + \frac{\vartheta'(\rho)}{\sqrt{p'(\rho)}} \right) (W - Z). \end{aligned} \right\} \tag{28}$$

其中

$$W = \frac{[p'(\rho)]^{-\frac{1}{4}}}{\rho^{\frac{1}{2}}} r_x + \frac{1}{\varepsilon} g(\rho), \quad Z = \frac{[p'(\rho)]^{-\frac{1}{4}}}{\rho^{\frac{1}{2}}} s_x + \frac{1}{\varepsilon} g(\rho). \tag{29}$$

由假设条件(P) 和(SC), 有

$$\frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} > 0, \quad 1 \pm \frac{\vartheta'(\rho)}{\sqrt{p'(\rho)}} > 0. \tag{30}$$

那么, 由式(28), (30)得

$$\left. \begin{aligned} D_{\lambda} W &\leq \frac{1}{2\varepsilon} \left( 1 - \frac{\theta'(\rho)}{\sqrt{p'(\rho)}} \right) (Z - W), \\ D_{\mu} Z &\leq \frac{1}{2\varepsilon} \left( 1 + \frac{\theta'(\rho)}{\sqrt{p'(\rho)}} \right) (W - Z). \end{aligned} \right\} \quad (31)$$

援用文[1]引理 2.3 的最大值原理的证法, 得

$$W(x, t), Z(x, t) \leq \max\{W(x, 0), Z(x, 0)\}.$$

因为

$$W(x, 0) = \frac{[p'(\rho_0(x))]^{\frac{1}{4}}}{(\rho_0(x))^{\frac{1}{2}}} r_0(x) + \frac{1}{\varepsilon} g(\rho_0(x)) \leq \frac{1}{\varepsilon} \frac{[p'(\rho^*)]^{\frac{1}{4}}}{\rho_*^{\frac{1}{2}}} M_1 = \frac{M_2}{\varepsilon}.$$

类似可得  $Z(x, 0) \leq \frac{M_2}{\varepsilon}$ . 所以

$$W(x, t) \leq \frac{M_2}{\varepsilon}, \quad Z(x, t) \leq \frac{M_2}{\varepsilon}. \quad (33)$$

由式(27), (29), 我们有

$$\left. \begin{aligned} W(x, t) &= D_{\mu} g(\rho(x, t)) + \frac{1}{\varepsilon} g(\rho(x, t)), \\ Z(x, t) &= D_{\lambda} g(\rho(x, t)) + \frac{1}{\varepsilon} g(\rho(x, t)). \end{aligned} \right\} \quad (34)$$

再由式(33), (34), 我们得到  $g(\rho(x, t)) \leq M_2$ . 引理 3 证毕.

**推论 1** 在假设条件(P)和条件(SC)下, 如果初值满足条件(I), 则柯西问题(1), (2)的光滑解  $(\rho(x, t), u(x, t))$  有如下估计式

$$0 < \rho \leq \rho(x, t) \leq \bar{\rho} \quad (35)$$

$$|u(x, t)| \leq M. \quad (36)$$

其中,  $\rho = \rho_* \exp\{-\frac{1}{\rho_*^2}(\rho'_*)^{\frac{1}{4}}M_2\}$ ,  $M, M_2$  分别由引理 1 和引理 3 所给定,  $\bar{\rho}$  由  $\int_0^{\bar{\rho}} \frac{\sqrt{\rho(\tau)}}{\tau} d\tau = M_0$  所给定.

从式(35)可知, 先验假设(H)是合理的. 同时表明, 如果初值离开真空状态, 则柯西问题(1), (2)的光滑解必一致地(关于时间  $t$ )离开真空状态.

**引理 4** 在假设条件(P), (SC)和(I)下, 如果存在  $\bar{\rho} \in \mathbf{R}^+$ , 使  $\delta = \sup_{\rho \in [\bar{\rho}, \bar{\rho}] |f(\rho) - f(\bar{\rho})|}$  充分小, 并且

$$|r'_0(x)| \leq \frac{M_3}{\varepsilon}, \quad |s'_0(x)| \leq \frac{M_3}{\varepsilon}. \quad (37)$$

则当  $M_3$  充分小时, 柯西问题(5), (6)的光滑解在其存在区域上有估计式

$$|r_x(x, t)| \leq \frac{M_4}{\varepsilon}, \quad |s_x(x, t)| \leq \frac{M_4}{\varepsilon}, \quad (38)$$

其中  $\delta, M_3, M_4$  为与  $\varepsilon$  无关的正常数.

**证明** 令

$$\left. \begin{aligned} F(x, t) &= [p'(\rho(x, t))]^{\frac{1}{2}} r_x(x, t) + \frac{1}{\varepsilon} [f(\rho(x, t)) - f(\bar{\rho})], \\ G(x, t) &= [p'(\rho(x, t))]^{\frac{1}{2}} s_x(x, t) - \frac{1}{\varepsilon} [f(\rho(x, t)) - f(\bar{\rho})]. \end{aligned} \right\} \quad (39)$$

根据式(26), (27), 经过计算可得

$$\left. \begin{aligned} F_t + \lambda F_x &= -A_+(\rho, F)(F + G), \\ G_t + \mu G_x &= -A_-(\rho, G)(F + G). \end{aligned} \right\} \quad (40)$$

其中

$$A_+(\rho, F) = \frac{1}{2\varepsilon} \left[ 1 + \frac{\rho'(\rho)}{\sqrt{p'(\rho)}} - \left( \frac{\rho''(\rho)}{2p'(\rho)} + 1 \right) \frac{f(\rho) - f(\bar{\rho})}{\sqrt{p'(\rho)}} \right] + \left[ \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right] \frac{F}{\sqrt{p'(\rho)}},$$
$$A_-(\rho, G) = \frac{1}{2\varepsilon} \left[ 1 - \frac{\rho'(\rho)}{\sqrt{p'(\rho)}} + \left( \frac{\rho''(\rho)}{2p'(\rho)} + 1 \right) \frac{f(\rho) - f(\bar{\rho})}{\sqrt{p'(\rho)}} \right] + \left[ \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right] \frac{G}{\sqrt{p'(\rho)}},$$

仿照文[1]引理 2.5 的证法, 令

$$A_{\pm} = \inf_{\rho \leq \bar{\rho}} \left[ 1 \pm \frac{\rho'(\rho)}{\sqrt{p'(\rho)}} \mp \left( \frac{\rho''(\rho)}{2p'(\rho)} + 1 \right) \frac{f(\rho) - f(\bar{\rho})}{\sqrt{p'(\rho)}} \right],$$
$$B_1 = \sup_{\rho \leq \bar{\rho}} \left[ \frac{\rho''(\rho)}{4p'(\rho)} + \frac{1}{2} \right] \frac{1}{\sqrt{p'(\rho)}},$$
$$B_2 = \sup_{\rho \leq \bar{\rho}} \sqrt{p'(\rho)}.$$

由次特征条件 (SC) 及引理所假设  $\delta$  充分小, 则  $A_{\pm} > 0$ . 仿照文[1]的引理 2.5 的证法, 由式(40) 得到估计式(38) 成立, 即引理 4 得证.

3 整体光滑解的存在性

根据文[6]关于一阶拟线性双曲组的柯西问题的局部光滑解的存在唯一性定理, 并由引理 1, 3, 4 和推论 1, 我们得到如下的主要结果.

定理 假设条件(P), (SC) 成立, 并假设存在与  $\varepsilon$  无关的正常数  $\rho_*$ ,  $\rho^*$ ,  $M'$ ,  $M''$  和  $\delta$ , 使得

$$\rho_* \leq \rho_0(x) \leq \rho^*, \quad |u_0(x)| \leq M', \quad |r_0(x)| \leq \frac{M''}{\varepsilon}, \quad |s_0(x)| \leq \frac{M''}{\varepsilon}.$$

并且存在  $\bar{\rho} \in \mathbf{R}^+$ , 使  $\delta = \sup_{\rho \leq \bar{\rho}} |f(\rho) - f(\bar{\rho})|$ ,  $\bar{\rho}$ ,  $\rho$  分别为光滑解  $\rho(x, t)$  的上界和下界. 如果  $M''$  和  $\delta$  充分小, 则柯西问题(1), (2) 存在唯一的整体光滑解  $(\rho(x, t), u(x, t))$ .

参 考 文 献

1 Yang Tong, Zhu Changjiang. Existence and non existence of global smooth solutions for p-system with relaxation[J]. J Giff. Equis., 2000. 161(2): 321~ 336

2 Zhao Huijiang. Nonlinear stability of strong planar rarefaction waves for the relaxation approximation of conservation laws in several space dimensions[J]. J Diff. Equis., 2000, 163(1): 198~ 222

3 Natalini R. Convergence to equilibrium for the relaxation approximations of conservation laws[J]. Comm. Pure Appl. Math., 1996, 49(4): 795~ 823

4 Li Caizhong, Liu Fagui. P-systems with relaxation[J]. Acta Math. Appl. Sina., 1997, 20(1): 61~ 69

5 伍锦棠, 郑永树. 带非线性松驰项的半线性双曲组的整体光滑解[J]. 华侨大学学报(自然科学版), 2003, 24(2): 131~ 135

6 Douglis A. Existence theorems for hyperbolic systems[J]. Comm. Pure Appl. Math., 1952, 5(2): 119~ 154

7 郑永树, 连碧龙. 欧拉坐标的气动力学方程组的整体光滑解[J]. 华侨大学学报(自然科学版), 2000, 21(4): 337~ 343

Globally Smoothing Solution to Euler's Equations with  
Relaxation Term  
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**Abstract** A study is made on Cauchy problem of compressible Euler's equations with relaxation term. Taking pressure function and sub characteristic condition as facts without proof, the globally smoothing solution to Cauchy problem of Euler's equations is proved to be uniquely existed if initial value is in a small  $C^1$  norm and initial density is away from state of vacuum.

**Keywords** aerodynamic equations, relaxation term, globally smoothing solution