第 24 卷　第 2 期　　华侨大学学报(自然科学版)　　Vol. 24　No. 2
2003 年 4 月　　Journal of Huaqiao University (Natural Science)　　Apr. 2003

文章编号　1000-5013(2003)02-0131-05

带非线性松弛项的半线性双曲组的整体光滑解

伍锦棠　　郑永树

(华侨大学数学系, 福建 泉州 362011)

摘要　研究一类带非线性松弛项的半线性双曲组的柯西问题, 对 C 模有界的初值, 证明其存在唯一的整体光滑解.

关键词　半线性双曲组, 松弛, 柯西问题, 整体光滑解

中图分类号　O 175.29　　　文献标识码　A

1　问题的提出

松弛现象在物理学中有重要作用. 如在单原子气体动力学理论中, 当平衡态受到干扰时, 它能逐渐松弛到带有麦克斯韦速度分布的平衡态. 在多原子气体的连续性理论中, 除了瞬间能量在气体受干扰时能很快调节到平衡态时的值外, 其他形式的能量通过气体分子间的相互碰撞松弛到它们平衡态下的值, 但其松弛时间并不短. 因此, 研究热的非平衡态变得很重要. 其它诸多松弛现象出现在交通流、带记忆的弹性物质等上. 对趋于平衡态的松弛过程, 可通过一个简单的模型进行研究. 1987 年, 针对松弛现象及平衡态问题, Liu 在文[1]中首先研究了具松弛项的拟线性双曲型方程组问题, 并给出解的非线性稳定的次特征条件. 1995 年, Jin 和 Xin 在文[2]中研究了一阶守恒律方程的柯西问题. 即

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0, && (x,t)\in \mathbf{R}\times\mathbf{R}^+,\\ &u(x,0)=u_0(x), && x\in\mathbf{R}.\end{aligned}\right\}\tag{1}$$

引进了带松弛项的半线性双曲组为

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=0,\\ &\frac{\partial v}{\partial t}+a^2\frac{\partial u}{\partial x}=-\frac{1}{\epsilon}(v-f(u)).\end{aligned}\right\}\tag{2}$$

其中 a, ϵ 为正常数. 在满足次特征条件(即 $-a\le f'(u)\le a$), Jin 和 Xin 研究了方程组(2)与方程(1)在平衡态下解的逼近关系. 对于具松弛项的双曲组柯西问题的整体连续解或光滑解的存在唯一性和解的渐近稳定性, 已有许多研究结果[3~10]. 本文对带非线性松驰项的半线性双

收稿日期　2002-09-19　　作者简介　伍锦棠(1977-), 男, 硕士研究生

曲组柯西问题的整体光滑解进行研究.

2 主要结果

本文研究带非线性松弛项的半线性双曲组的柯西问题:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial v}{\partial x}=0, \qquad (x,t)\in \mathbf{R}\times\mathbf{R}^{+},\\ &\frac{\partial v}{\partial t}+a^{2}\frac{\partial u}{\partial x}=-\frac{1}{\epsilon}(g(v)-f(u)), \qquad (x,t)\in \mathbf{R}\times\mathbf{R}^{+}.\end{aligned}\right\} \tag{3}$$

$$(u,v)(x,0)=(u_0(x),v_0(x)), \qquad x\in\mathbf{R}. \tag{4}$$

其中a, ϵ为正常数.

假设函数$f(u)$, $g(v)$应满足条件为

$$f(u),g(v)\in C^{1}, \qquad g'(v)\geq 1, \qquad f(0)=g(0)=0. \tag{H$_1$}$$

式(3)应满足次特征条件为

$$\sup_{|u|\leq(|u_0|_{C^0}+\frac{1}{a}|v_0|_{C^0})}\left|f'(u)\right|<a. \tag{S}$$

其中$|u_0|_{C^0}=\sup_{x\in\mathbf{R}}|u_0(x)|$, $|v_0|_{C^0}=\sup_{x\in\mathbf{R}}|v_0(x)|$. 本文有如下主要结果.

定理 如果初值$u_0(x),v_0(x)\in C_b^1(\mathbf{R})$, 在满足假设(H$_1$)及次特征条件(S)下, 柯西问题(3),(4)在$t\geq 0$的上半平面存在唯一的光滑解.

3 定理的证明

为简单起见, 不妨令$\epsilon=1$, 若不然只需作变换$(x,t)\to(\epsilon x,\epsilon t)$. 在证明定理之前, 我们先给出解的先验估计.

引理1 在定理条件满足的前提下, 柯西问题(3),(4)的解$(u(x,t),v(x,t))$的C^0模的估计为

$$\left.\begin{aligned} &|u(x,t)|\leq|u_0|_{C^0}+\frac{1}{a}|v_0|_{C^0},\\ &|v(x,t)|\leq 2(a|u_0|_{C^0}+|v_0|_{C^0}).\end{aligned}\right\} \tag{5}$$

证明 引进黎曼不变量

$$r=-v+au, \quad s=v+au. \tag{6}$$

借助式(6),方程组(3)可化为

$$\left.\begin{aligned} &r_t-ar_x=-f\left(\frac{r+s}{2a}\right)+g\left(\frac{s-r}{2}\right),\\ &s_t+as_x=f\left(\frac{r+s}{2a}\right)-g\left(\frac{s-r}{2}\right).\end{aligned}\right\} \tag{7}$$

设

$$1\leq g'(v)\leq K. \tag{8}$$

令

$$\bar{r}=re^{Kt}, \qquad \bar{s}=se^{Kt}. \tag{9}$$

则有

$$\left.\begin{aligned}\bar r_t - a\bar r_x &= -f\left(\frac{r+s}{2a}\right)e^{Kt} + g\left(\frac{s-r}{2}\right)e^{Kt} + K\bar r,\\ \bar s_t + a\bar s_x &= f\left(\frac{r+s}{2a}\right)e^{Kt} - g\left(\frac{s-r}{2}\right)e^{Kt} + K\bar s.\end{aligned}\right\} \tag{10}$$

设 $x_1 < x$. 用 $\zeta_+ = x_1 - a(t-\tau)$, $\zeta_- = x_2 + a(t-\tau)$ 表示, 方程组(9)过点(x,t)的两条特征线. 分别沿上述两特征线积分方程组(10)中两式, 得

$$\left.\begin{aligned}\bar r(x_2,t) &= r_0(x_2+at) + \int_0^t\left[-f\left(\frac{r_-+s_-}{2a}\right)e^{K\tau} + g\left(\frac{s_--r_-}{2}\right)e^{K\tau} + K\bar r_-\right]d\tau,\\ \bar s(x_1,t) &= s_0(x_1-at) + \int_0^t\left[f\left(\frac{r_++s_+}{2a}\right)e^{K\tau} - g\left(\frac{s_+-r_+}{2}\right)e^{K\tau} + K\bar s_+\right]d\tau,\end{aligned}\right\} \tag{11}$$

其中$\bar r_\pm = \bar r(\zeta_\pm,\tau)$, $\bar s_\pm = \bar s(\zeta_\pm,\tau)$. 方程组(11)中两式相加, 并利用$(H_1)$及微分中值定理, 得

$$\begin{aligned}\bar r(x_2,t) + \bar s(x_1,t) = r_0(x_2+at) + s_0(x_1-at) + \\ \int_0^t\left\{\left(\frac{g'(\zeta_1)}{2} + \frac{f'(\zeta_1)}{2a}\right)(\bar r_+ + \bar s_+) + \left(\frac{g'(\zeta_1)}{2} - \frac{f'(\zeta_1)}{2a}\right)(\bar r_- + \bar s_-) + \right.\\ \left.(K - g'(\zeta_2))(\bar r_- + \bar s_+)\right\}d\tau,\end{aligned} \tag{12}$$

令

$$R(t) = \sup_{-\infty<x_1\le x_2<+\infty}\left|\bar r(x_2,t) + \bar s(x_1,t)\right|. \tag{13}$$

由式(12)和式(13), 显然有

$$R(t) \le \left|r_0\right|_{C^0} + \left|s_0\right|_{C^0} + K\int_0^t R(\tau)\,d\tau.$$

易得

$$R(t) \le (\left|r_0\right|_{C^0} + \left|s_0\right|_{C^0})e^{Kt}. \tag{14}$$

根据式(9), (13), (14), 即有

$$\left|r(x,t) + s(x,t)\right| \le \left|r_0\right|_{C^0} + \left|s_0\right|_{C^0}. \tag{15}$$

另令

$$\left.\begin{aligned}\tilde r &= r e^{\int_0^t g'(\zeta_2)(\zeta_-,\tau)d\tau},\\ \tilde s &= s e^{\int_0^t g'(\zeta_2)(\zeta_+,\tau)d\tau}.\end{aligned}\right\} \tag{16}$$

由式(H_1), (7), (16), 易知有

$$D_-\tilde r = \left(\frac{g'(\zeta_2)}{2} - \frac{f'(\zeta_1)}{2a}\right)(\tilde r + \tilde s). \tag{17}$$

其中$D_\pm = \frac{\partial}{\partial t} \pm a\frac{\partial}{\partial x}$. 沿特征线$\zeta = x + a(t-\tau)$积分式(17), 得

$$\tilde r(x,t) = r_0(x+at) + \int_0^t\left(\frac{g'(\zeta_2)}{2} - \frac{f'(\zeta_1)}{2a}\right)(\tilde r + \tilde s)(x + a(t-\tau),\tau)\,d\tau. \tag{18}$$

由式(H_1), (15), (16), (18), 易得

$$\left|r(x,t)\right| \le \left|r_0\right|_{C^0} + \left|s_0\right|_{C^0}. \tag{19}$$

同理有

$$\left|s(x,t)\right| \le \left|r_0\right|_{C^0} + \left|s_0\right|_{C^0}. \tag{20}$$

由式(6), (19), (20), 显然有式(5)成立. 引理 1 得证.

命题 柯西问题为

$$\begin{cases} \dfrac{\partial u}{\partial t}+\lambda(x,t)\dfrac{\partial u}{\partial x}=a(x,t)(v-u), & (x,t)\in \mathbf{R}\times\mathbf{R}^{+}, \\ \dfrac{\partial v}{\partial t}+\mu(x,t)\dfrac{\partial v}{\partial x}=b(x,t)(u-v), & (x,t)\in \mathbf{R}\times\mathbf{R}^{+}, \end{cases}$$

$$(u,v)(x,0)=(u_0(x),v_0(x)),\qquad x\in\mathbf{R}.$$

在上式中, 函数 $a(x,t)$, $b(x,t)$ 为非负连续有界, $\lambda(x,t)$, $\mu(x,t)$ 有界. 若

$$m\le u_0(x),\qquad v_0(x)\le M,\qquad x\in\mathbf{R},$$

则

$$m\le u(x,t),\qquad v(x,t)\le M.$$

此命题, 可用文〔1〕中的极值原理予以证明.

引理 **2** 在定理的假设条件下, 柯西问题(3), (4) 的解$(u(x,t),v(x,t))$, 其一阶偏导数的估计为

$$\left.\begin{aligned} |u_x(x,t)| &\le \frac{1}{a}\left(a|u_0'(x)|_{C^0}+|v_0'(x)|_{C^0}+2\bar{M}\right), \\ |v_x(x,t)| &\le a|u_0'(x)|_{C^0}+|v_0'(x)|_{C^0}+2\bar{M}. \end{aligned}\right\} \tag{21}$$

其中 $\bar{M}=\sup\limits_{|u|\le(|u_0|_{C^0}+\frac{1}{a}|v_0|_{C^0})}|f(u)|$.

证明 方程组(3) 两边关于 x 求导, 得

$$\left.\begin{aligned} &(u_x)_t+(v_x)_x=0, \\ &(v_x)_t+a^2(u_x)_x=f'(u)u_x-g'(v)v_x. \end{aligned}\right\} \tag{22}$$

引进黎曼不变量

$$w=-v_x+au_x,\quad z=v_x+au_x. \tag{23}$$

方程组(22) 化为

$$\left.\begin{aligned} w_t-aw_x &= -\left(\frac{g'(v)}{2}+\frac{f'(u)}{2a}\right)w+\left(\frac{g'(v)}{2}-\frac{f'(u)}{2a}\right)z, \\ z_t+az_x &= \left(\frac{g'(v)}{2}+\frac{f'(u)}{2a}\right)w-\left(\frac{g'(v)}{2}-\frac{f'(u)}{2a}\right)z. \end{aligned}\right\} \tag{24}$$

令

$$\bar{w}=w-\frac{f(u)}{a},\quad \bar{z}=z-\frac{f(u)}{a}. \tag{25}$$

由式(24), (25), 得

$$\left.\begin{aligned} D_-\bar{w} &= \left(\frac{g'(v)}{2}+\frac{f'(u)}{2a}\right)(\bar{z}-\bar{w}), \\ D_+\bar{z} &= \left(\frac{g'(v)}{2}-\frac{f'(u)}{2a}\right)(\bar{w}-\bar{z}). \end{aligned}\right\}$$

由式(H_1), (S) 及引理 1 可得到 $\dfrac{g'(v)}{2}\pm\dfrac{f'(u)}{2a}\ge 0$, 并且连续有界. 故由命题, 可得

$$|\bar{w}(x,t)|,|\bar{z}(x,t)|\le \max(|\bar{w}_0|_{C^0},|\bar{z}_0|_{C^0}). \tag{26}$$

显然, 由式(23), (25), (26), 易得式(21). 引理 2 得证.

定理的证明 在定理的假设条件下, 根据文〔2〕, 柯西问题(3), (4) 存在唯一的局部光

滑解. 又由引理 1 及引理 2 的先验估计结果和解的延拓定理, 可以得到在 $t \geq 0$ 的上半平面存在唯一的光滑解. 故定理得证.

参　考　文　献

1 Liu Taiping. Hyperbolic conservation laws with relaxation[J]. Comm. Math. Phy., 1987, 108: 153～175

2 Jin Shi, Xin Zhouping. The relaxation scheme for systems of conservation laws in arbitrary space dimensions[J]. Comm. Pure Appl. Math., 1995, 48: 235～276

3 Zhu Changjiang, Yang Tong. Existence and non-existence of global smooth solutions for P-system with relaxation[J]. J. D. E., 2000, 161: 321～336

4 Li Caizhong, Liu Fagui. P-systems with relaxation[J]. Acta. Math. Appl. Sina., 1997, 20: 61～69

5 Zhao Huijiang. Nonlinear stability of strong planar rarefaction waves for the relaxation approximation of conservation laws in several space dimensions[J]. J. D. E., 2000, 163: 198～222

6 Roberto Natalini. Covergence to equilibrium for the relaxation approximations of conservation laws[J]. Comm. Pure Appl. Math., 1996, 49: 795～823

7 Luo Tao. Asymptotic stability of planer rarefaction waves for the relaxation approximation of conservation laws in several dimensions[J]. J.D.E., 1997, 133: 255～279

8 Matsumura A, Nishihara K. Global stability of the rarefaction waves of a one-dimensional modelsystem for compressible visous gas[J]. Comm. Math. Phys., 1992, 144: 325～335

9 Corrado M, Roberto N. L^1 nolinear stability of traveling waves for a hyperbolic system with relaxation [J]. J. D. E., 1996, 132: 275～292

10 Liu Hailiang. The L^p stability of relaxation rarefaction profiles[J]. J. D. E., 2001, 171: 397～411

11 Bressan A, Marson A. Maximum principle for optimally controlled systems of conservation laws[J]. Rend. Sem. Mat. Univ. Padova., 1995, 94: 74～94

12 Douglis A. Existence theorems for hyperbolic systems[J]. Comm. Pure Appl. Math., 1952, 5: 119～154

Globally Smooth Solution to a Semi-Linear Hyperbolic System with a Nonlinear Relaxation Term

Wu Jintang　　Zheng Yongshu

(Dept. of Math., Huaqiao Univ., 362011, Quanzhou, China)

Abstract　A study is made on the Cauchy problem of a class of semi-linear hyperbolic system with a nonlinear relaxation term. The globally smooth solution is proved to be uniquely existed in the initial value with bounded C^1 norm.

Keywords　semi-linear hyperbolic system, relaxation, Cauchy problem, globally smooth solution