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解四阶抛物型方程的高精度差分格式

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摘要　对四阶抛物型方程构造了一族含参数高精度三层差分格式. 当参数满足一定的条件时, 差分格式稳定, 局部截断误差阶数最高可达 $O(\tau^2 + h^6)$. 最后, 用数值例子说明对稳定性所作的分析是正确的.

关键词　四阶抛物型方程, 差分格式, 绝对稳定
中图分类号　O 241.82　　文献标识码　A

考虑下列四阶抛物型方程初边值问题

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0, \qquad 0 < x < 1, \quad 0 < t \leq T, \\ &u(x, 0) = f(x), \qquad 0 \leq x \leq 1, \\ &u(0, t) = \frac{\partial^2 u(0, t)}{\partial x^2} = u(1, t) = \frac{\partial^2 u(1, t)}{\partial x^2} = 0, \qquad 0 \leq t \leq T. \end{aligned}\right\} \tag{1}$$

1960 年, Å½Ì¶ 对四阶抛物型方程(1)构造了两个隐式差分格式[1], 其截断误差分别为 $O(\tau^2 + h^2)$ 及 $O(\tau^2 + h^4)$. 其他学者也对四阶抛物形方程进行了研究[2,3], 得到了不同的结果, 其中有的截断误差阶达到了 $O(\tau^2 + h^6)$. 本文构造了一族新的三层含参数隐式格式, 当诸参数满足定理 的条件时, 所给出的格式绝对稳定. 同时, 还得到一个显式格式. 它们的截断误差阶数均为 $O(\tau^2 + h^6)$.

1　差分格式的构造

以下分别用 τ, h 表示时间 t 及空间方向 x 的步长, 用 u_j^n 表示 $u(jh, n\tau)$ 的差分逼近. 网域由点集 (x_j, t_n) $(j = 0, 1, 2, \cdots, J; n = 0, 1, 2, \cdots)$ 所组成, 其中 $x_j = jh$, $t_n = n\tau$, $h = 1/J$. 又设 $r = \tau/h^4$ 为网格比. 初边界条件的离散化处理同文献[1].

用如下含参数并具有对称形式的差分方程逼近微分方程(1), 可得

$$\theta_{12}\frac{u_{j-2}^{n+1} - u_{j-2}^{n}}{\tau} + \theta_{11}\frac{u_{j-1}^{n+1} - u_{j-1}^{n}}{\tau} + \theta_{10}\frac{u_{j}^{n+1} - u_{j}^{n}}{\tau} + \theta_{11}\frac{u_{j+1}^{n+1} - u_{j+1}^{n}}{\tau} +$$

$$\theta_{12}\frac{u_{j+2}^{n+1} - u_{j+2}^{n}}{\tau} + \theta_{22}\frac{u_{j-2}^{n} - u_{j-2}^{n-1}}{\tau} + \theta_{21}\frac{u_{j-1}^{n} - u_{j-1}^{n-1}}{\tau} + \theta_{20}\frac{u_{j}^{n} - u_{j}^{n-1}}{\tau} +$$

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$$\theta_{21}\frac{u_{j+1}^{n}-u_{j+1}^{n-1}}{\tau}+\theta_{22}\frac{u_{j+2}^{n}-u_{j+2}^{n-1}}{\tau}+\theta_{3}\frac{\delta_x^4 u_j^{n+1}}{h^4}+\theta_{4}\frac{\delta_x^4 u_j^{n}}{h^4}+\theta_{5}\frac{\delta_x^4 u_j^{n-1}}{h^4}=0, \tag{2}$$

其中 $\theta_{12},\theta_{11},\theta_{10},\theta_{22},\theta_{21},\theta_{20},\theta_3,\theta_4,\theta_5$ 为待定参数. 适当选取这些参数, 可以使差分格式(2)逼近微分方程(1). 它具有尽可能高阶的离散误差, 而且有较好的稳定性. δ_x^4 表示 x 方向的四阶中心差分算子, 即 $\delta_x^4 u_j^n = u_{j-2}^n - 4u_{j-1}^n + 6u_j^n - 4u_{j+1}^n + u_{j+2}^n$. 当微分方程(1)的解充分光滑时, 有如下的关系式成立, 即

$$\frac{\partial^{p+q}u}{\partial x^p \partial t^q} = (-1)^q \frac{\partial^{p+4q}u}{\partial x^{p+4q}}, \qquad p,q = 0,1,2,3,\cdots. \tag{3}$$

将格式(2)中各节点上的 u 在网点(x_j,t_n)或$(jh,n\tau)$处展开的 Taylor 级数代入, 经整理后得

$$(\theta_{10}+2\theta_{11}+2\theta_{12}+\theta_{20}+2\theta_{21}+2\theta_{22})\frac{\partial u}{\partial t}+(\theta_3+\theta_4+\theta_5)\frac{\partial^4 u}{\partial x^4}+\tau\Big[\frac{1}{2}(\theta_{10}+2\theta_{11}+2\theta_{12}-\theta_{20}-2\theta_{21}-2\theta_{22})\frac{\partial^2 u}{\partial t^2}+(\theta_3-\theta_5)\frac{\partial^5 u}{\partial x^4\partial t}\Big]+\tau^2\Big[\frac{1}{3!}(\theta_{10}+2\theta_{11}+2\theta_{12}+\theta_{20}+2\theta_{21}+2\theta_{22})\frac{\partial^3 u}{\partial t^3}+\frac{1}{2}(\theta_3+\theta_5)\frac{\partial^6 u}{\partial x^4\partial t^2}\Big]+h^2\Big[\frac{6}{3!}(\theta_{11}+4\theta_{12}+\theta_{21}+4\theta_{22})\frac{\partial^3 u}{\partial x^2\partial t}+\frac{1}{6}(\theta_3+\theta_4+\theta_5)\frac{\partial^6 u}{\partial x^6}\Big]+\tau h^2\Big[\frac{1}{2}(\theta_{11}+4\theta_{12}-\theta_{21}-4\theta_{22})\frac{\partial^4 u}{\partial x^2\partial t^2}+\frac{1}{6}(\theta_3-\theta_5)\frac{\partial^7 u}{\partial x^6\partial t}\Big]+$$

$$\tau^2h^2\Big[\frac{20}{5!}(\theta_{11}+4\theta_{12}+\theta_{21}+4\theta_{22})\frac{\partial^5 u}{\partial x^2\partial t^3}+\frac{1}{12}(\theta_3+\theta_5)\frac{\partial^8 u}{\partial x^6\partial t^2}\Big]+h^4\Big[\frac{10}{5!}(\theta_{11}+16\theta_{12}+\theta_{21}+16\theta_{22})\frac{\partial^5 u}{\partial x^4\partial t}+\frac{1}{80}(\theta_3+\theta_4+\theta_5)\frac{\partial^8 u}{\partial x^8}\Big]+\tau h^4\Big[\frac{30}{6!}(\theta_{11}+16\theta_{12}-\theta_{21}-16\theta_{22})\frac{\partial^6 u}{\partial x^2\partial t^2}+\frac{1}{80}(\theta_3-\theta_5)\frac{\partial^9 u}{\partial x^8\partial t}\Big]+\tau^3h^2\Big[\frac{30}{6!}(\theta_{11}+4\theta_{12}-\theta_{21}-4\theta_{22})\frac{\partial^6 u}{\partial x^2\partial t^4}+\frac{1}{84}(\theta_3-\theta_5)\frac{\partial^9 u}{\partial x^6\partial t^3}\Big]+$$

$$\tau^2h^4\Big[\frac{70}{7!}(\theta_{11}+16\theta_{12}+\theta_{21}+16\theta_{22})\frac{\partial^7 u}{\partial x^4\partial t^3}+\frac{1}{160}(\theta_3+\theta_5)\frac{\partial^{10} u}{\partial x^8\partial t^2}\Big]+\tau^4h^2\Big[\frac{42}{7!}(\theta_{11}+4\theta_{12}+\theta_{21}+4\theta_{22})\frac{\partial^7 u}{\partial x^2\partial t^5}+\frac{1}{16\times 9}(\theta_3+\theta_5)\frac{\partial^{10} u}{\partial x^6\partial t^4}\Big]+h^6\Big[\frac{14}{7!}(\theta_{11}+2^6\theta_{12}+\theta_{21}+2^6\theta_{22})\frac{\partial^7 u}{\partial x^6\partial t}+\frac{17}{16\times 1\,890}(\theta_3+\theta_4+\theta_5)\frac{\partial^{10} u}{\partial x^{10}}\Big]+O(\tau^3+\tau h^5+h^7)=0. \tag{4}$$

利用关系式(3), 当下列诸条件同时成立时, 差分格式(2)的截断误差为 $O(\tau^2+h^6)$.

$$\left.\begin{aligned}&\theta_{10}+2\theta_{11}+2\theta_{12}+\theta_{20}+2\theta_{21}+2\theta_{22}=1,\\&\theta_3+\theta_4+\theta_5=1,\\&\theta_{10}+2\theta_{11}+2\theta_{12}-\theta_{20}-2\theta_{21}-2\theta_{22}=2(\theta_3-\theta_5),\\&6(\theta_{11}+4\theta_{12}+\theta_{21}+4\theta_{22})=1,\\&20(\theta_{11}+16\theta_{12}+\theta_{21}+16\theta_{22})=3.\end{aligned}\right\} \tag{5}$$

解方程组(5)得

$$\theta_{20}=\frac{79}{2^3\times 3\times 5}-\theta_0,\quad \theta_{10}=\theta_0, \tag{6a}$$

$$\left.\begin{aligned}&\theta_{21}=\frac{31}{2^2\times 3^2\times 5}-\theta_1,\quad \theta_{11}=\theta_1,\\&\theta_{22}=\frac{-1}{2^4\times 3^2\times 5}-\theta_2,\quad \theta_{12}=\theta_2,\\&\theta_4=\frac{1}{2}+(\theta_0+2\theta_1+2\theta_2)-2\theta_3,\\&\theta_5=\frac{1}{2}-(\theta_0+2\theta_1+2\theta_2)+\theta_3.\end{aligned}\right\}\tag{6b}$$

将式(6)代入差分格式(2)，可得如下含参数($\theta_0, \theta_1, \theta_2, \theta_3$)的三层隐格式为

$$\theta_2(u_{j-2}^{n+1}+u_{j+2}^{n+1})+\theta_1(u_{j-1}^{n+1}+u_{j+1}^{n+1})+\theta_0 u_j^{n+1}+\theta_3 r\delta_x^4 u_j^{n+1}=$$
$$\left(\frac{1}{720}+2\theta_2\right)(u_{j-2}^n+u_{j+2}^n)-\left(\frac{31}{180}-2\theta_1\right)(u_{j-1}^n+u_{j+1}^n)-\left(\frac{79}{120}-2\theta_0\right)u_j^n-$$
$$\left[\frac{1}{2}+(\theta_0+2\theta_1+2\theta_2)-2\theta_3\right]r\delta_x^4u_j^n-\left(\frac{1}{720}+\theta_2\right)(u_{j-2}^{n-1}+u_{j+2}^{n-1})+\left(\frac{31}{180}-\theta_1\right)(u_{j-1}^{n-1}+$$
$$u_{j+1}^{n-1})+\left(\frac{79}{120}-\theta_0\right)u_j^{n-1}-\left[\frac{1}{2}-(\theta_0+2\theta_1+2\theta_2)+\theta_3\right]r\delta_x^4u_j^{n-1},\tag{7}$$

其截断误差为 $O(\tau^2+h^6)$. 当参数($\theta_0, \theta_1, \theta_2, \theta_3$)取不同的值时，可得不同的差分格式.

(Ⅰ) 当参数 $\theta_0=1, \theta_1=\theta_2=\theta_3=0$ 时，式(7)为三层显格式

$$u_j^{n+1}=\frac{1}{720}(u_{j-2}^n+u_{j+2}^n)-\frac{31}{180}(u_{j-1}^n+u_{j+1}^n)+\left(\frac{161}{120}u_j^n-\frac{3}{2}r\delta_x^4u_j^n-\right.$$
$$\frac{1}{720}(u_{j-2}^{n-1}+u_{j+2}^{n-1})+\frac{31}{180}(u_{j-1}^{n-1}+u_{j+1}^{n-1})-\frac{41}{120}u_j^{n-1}+\frac{1}{2}r\delta_x^4u_j^{n-1},\tag{8}$$

其截断误差仍为 $O(\tau^2+h^6)$ 的差分格式.

(Ⅱ) 当参数 $\theta_0=\theta_1=\theta_2=\theta_3=0$，或者 $\theta_0=\frac{79}{120}, \theta_1=\frac{31}{180}, \theta_2=-\frac{1}{720}$，以及 $\theta_3=1$ 时，式(7)为文献[2]的两层十点格式，且具有同样的截断误差.

(Ⅲ) 类似地还可以得到其它不同的差分格式. 这里从略.

2　差分格式的稳定性

为了证明格式的稳定性需要引用以下引理.

引理　即 Miller 准则. 实系数二次方程为 $Ax^2+Bx+C=0\ (A>0)$，其两个根按模小于等于 1 的充要条件是 $A-C\geq 0$, $A+B+C\geq 0$, $A-B+C\geq 0$.

定理 **1**　格式(7)绝对稳定的一个充分条件为

$$\left.\begin{aligned}&\theta_2\geq -\frac{1}{1\,440},\\&\theta_3\geq\frac{1}{4},\\&\frac{1}{2}\leq\theta_0+2\theta_1+2\theta_2\leq 2\theta_3,\\&\theta_1+4\theta_2\leq\frac{1}{12},\end{aligned}\right\}\tag{9}$$

证明　由 Fourier 分析法[4]可知格式(7)的特征方程为

$$A\lambda^2 + B\lambda + C = 0, \tag{10}$$

其中

$$\left.\begin{aligned}
A &= 16(\theta_2 + r\theta_3)s^4 - 4(\theta_1 + 4\theta_2)s^2 + (\theta_0 + 2\theta_1 + 2\theta_2),\\
B &= 16[(\frac{1}{2} + \theta_0 + 2\theta_1 + 2\theta_2 - 2\theta_3)r - (2\theta_2 + \frac{1}{720}]s^4 + \\
&\quad [8(\theta_1 + 4\theta_2) - \frac{2}{3}]s^2 + [1 - 2(\theta_0 + 2\theta_1 + 2\theta_2)],\\
C &= 16\{(\theta_2 + \frac{1}{720}) + [\frac{1}{2} - (\theta_0 + 2\theta_1 + 2\theta_2) + \theta_3]r\}s^4 + \\
&\quad [\frac{2}{3} - 4(\theta_1 + 4\theta_2)]s^2 + (\theta_0 + 2\theta_1 + 2\theta_2 - 1).
\end{aligned}\right\} \tag{11}$$

当 $\theta_0 + 2\theta_1 + 2\theta_2 \geq \frac{1}{2}$ 时,有

$$A - C = 1 - \frac{2}{3}s^2 - \frac{1}{45}s^2 + 8(2\theta_0 + 4\theta_1 + 4\theta_2 - 1)rs^4 \geq 1 - \frac{2}{3} - \frac{1}{45} = \frac{14}{45} > 0,$$

$$A + B + C = 16rs^4 \geq 0.$$

当定理的条件(9)满足时,有

$$A - B + C = 2(2\theta_0 + 4\theta_1 + 4\theta_2 - 1) + (\frac{4}{3} - 16\theta_1 + 64\theta_2)s^2 +$$

$$\{(\frac{2}{45} + 64\theta_2) + 32[2\theta_3 - (\theta_0 + 2\theta_1 + 2\theta_2)]r\}s^4 \geq 0.$$

由引理知,特征方程(10)的特征根的模小于等于1.又因为 $A - C > 0$,故特征方程(10)无重根.即对于任意的 $r > 0$,格式(7)绝对稳定.定理得证.

定理 2　当 $r \leq \frac{1}{16}$ 时,格式(8)稳定.

证明　当参数 $\theta_0 = 1, \theta_1 = \theta_2 = \theta_3 = 0$ 时,有

$$A - C = 1 - \frac{2}{3}s^2 - \frac{1}{45}s^2 + 8rs^4 \geq 1 - \frac{2}{3} - \frac{1}{45} = \frac{14}{45} > 0,$$

$$A + B + C = 16rs^4 \geq 0.$$

当 $r \leq \frac{1}{16}$ 时,$A - B + C = 2 + \frac{4}{3}s^2 + (\frac{2}{45} - 32r)s^4 \geq 2 - 32r \geq 0$.定理得证.

3　数值例子

考虑下列四阶抛物型初边值问题

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} = 0, \qquad 0 < x < \pi, t > 0,\\
&u(x, 0) = \sin x, \qquad 0 \leq x \leq \pi,\\
&u(0, t) = \frac{\partial^2 u(0, t)}{\partial x^2} = u(\pi, t) = \frac{\partial^2 u(\pi, t)}{\partial x^2} = 0, \qquad t > 0,
\end{aligned}\right\} \tag{12}$$

其精确解为 $u(x, t) = e^{-t}\sin x$.

初始值以及边界条件处理同文献[2].

取 $h=\pi/32$, 而时间步长按 $\tau=r\cdot h^4$ 进行计算. 因本文所构造的格式为三层格式, 除初始层网格函数值为已知外, 还需要用其它方法预先计算出第 1 层网格上的函数值. 为简化计算, 第 1 层网格函数值按精确值进行计算.

下面给出精度比较表. 其值等于精确解减去差分格式解的数值, 计算结果如表 1 所示. 从表可以看出, 由定理 1 和定理 2 给出的稳定性条件, 仅仅是格式(7) 和(8) 稳定的一个充分条件. 不同的参数对格式稳定性的影响不同.

表 1　精度比较表($n=500$)

格式	r	θ_0	θ_1	θ_2	θ_3	x: $5\pi/32$	$13\pi/32$	$21\pi/32$	$29\pi/32$
(7)	1	1/2	1/1 440	$-$1/1 440	1/4	3.4×10^{-3}	6.9×10^{-3}	6.4×10^{-3}	2.1×10^{-3}
	1/16	1	$-$1/1 440	1/1 440	1	2.2×10^{-4}	4.5×10^{-4}	4.2×10^{-4}	1.3×10^{-4}
	1/16	1/2	$-$/1 440	1/1 440	1/4	4.8×10^{-3}	9.8×10^{3}	9.0×10^{3}	2.9×10^{3}
	1/16	1/3	$-$/1 400	1/1 400	1/4	1.0×10^{143}	3.0×10^{143}	3.0×10^{143}	1.0×10^{143}
(8)	1/16	1	0	0	0	1.2×10^{-11}	2.5×10^{-11}	2.3×10^{-11}	7.5×10^{-12}
	1/20					7.8×10^{-12}	1.6×10^{-11}	1.5×10^{-11}	4.8×10^{-12}
	1/10					3.2×10^{-11}	6.4×10^{-11}	6.0×10^{-11}	2.0×10^{-11}
	1/9					-3.4×10^{11}	-7.0×10^{11}	-6.4×10^{11}	-2.1×10^{11}

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Difference Schemes of High Accuracy for Solving Parabolic Equation of Four Order

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Abstract　A family of highly accurate and three layer difference schemes containing parameter are constructed for solving parabolic equation of four order. In case the parameter satisfies definite conditions, these difference schemes are stable, with the maximum order of local truncation error up to $O(\tau^2+h^6)$. The stability analysis is shown by numerical example to be correct.

Keywords　parabolic equation of four order, difference scheme, absolutely stable