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二维对流扩散方程恒稳的蛙跳积分格式

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摘要 给出二维对流扩散方程的单点精细积分法导出的显式蛙跳积分格式,并证明它是无条件稳定的. 进行相容性分析,给出相容性条件. 用数值例子,表明该格式是有效的.

关键词 二维对流扩散方程, 蛙跳积分格式, 无条件稳定, 相容性条件

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1 子域精细积分及有限差分积分的格式

用差分法解抛物型方程与双曲型方程,将微分算子离散化为差分会带来稳定性与精度问题. 这一点对多维情况尤甚.

将偏微分方程化简并不必对全部坐标都离散. 例如,对二维扩散方程

$$\frac{\partial u}{\partial t}=a\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\quad(0<x,y<1,t>0),\tag{1}$$

其中 a 为正的常数. 对函数 $u=u(x,y,t)$ 在 x, y 方向离散,为方便起见,设空间步长 $\Delta x=\Delta y=\frac{1}{M}$, $x_i=i\Delta x$, $y_j=j\Delta y$ $(i,j=0,1,\cdots,M)$. 于是,空间坐标离散化后导出常微分方程组为

$$\frac{\mathrm{d}u_{ij}}{\mathrm{d}t}=\frac{a}{(\Delta x)^2}\{(u_{i+1,j}-2u_{i,j}+u_{i-1,j})+(u_{i,j+1}-2u_{i,j}+u_{i,j-1})\}\quad(i,j=1,2,\cdots,M-1).\tag{2}$$

先推导最简单的 FTCS 格式. 其离散的时间可表为 $t_n=n\Delta t(n=0,1,\cdots)$. 若取$\frac{\mathrm{d}u_{ij}}{\mathrm{d}t}=\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}$, 则方程(2)导出了熟知的对方程(1)的有限差分 FTCS 格式[]

$$u_{i,j}^{n+1}=u_{i,j}^{n}+\frac{4a\Delta t}{(\Delta x)^2}\left\{\frac{u_{i+1,j}^{n}+u_{i-1,j}^{n}+u_{i,j+1}^{n}+u_{i,j-1}^{n}}{4}-u_{ij}^{n}\right\}.\tag{3}$$

其稳定性条件为 $r=a\Delta t/(\Delta x)^2\leq\frac{1}{4}$. 但随着方程维数的增加,稳定性条件也越加苛刻. 若空间维数为 p, 则稳定性条件为 $r=\frac{a\Delta t}{(\Delta x)^2}\leq\frac{1}{2p}$.

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但也可将方程(2)写为

$$\frac{du_{i,j}}{dt}+\frac{4a}{(\Delta x)^2}u_{i,j}=\frac{a}{(\Delta x)^2}\{u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\},$$

只要将在右端的 $u_{i\pm1,j}$ 及 $u_{i,j\pm1}$ 当成为 $u^n_{i\pm1,j}$ 及 $u^n_{i,j\pm1}$. 于是,单点子域精细积分给出 FTCS 积分格式[2]为

$$u^{n+1}_{i,j}=\left[u^n_{i,j}-\frac{u^n_{i+1,j}+u^n_{i-1,j}+u^n_{i,j+1}+u^n_{i,j-1}}{4}\right]e^{-4a\Delta t/(\Delta x)^2}+\frac{u^n_{i+1,j}+u^n_{i-1,j}+u^n_{i,j+1}+u^n_{i,j-1}}{4}. \tag{4}$$

若令 $\lambda=\exp(-4a\Delta t/(\Delta x)^2)$, 则格式(4)的传播因子为 $G=\lambda+\frac{1}{2}(1-\lambda)(\cos p\pi\Delta x+\cos q\pi\Delta y)$, 其中 p, q 为任意实参数. 注意到 $\Delta t>0$, $a>0$, 则 $0<\lambda\leq1$. 又 $|\cos q\pi\Delta x|\leq1$, $|\cos q\pi\Delta y|\leq1$, 于是 $|G|\leq1$. 由 Von Neumann 法[1], 知 FECS 积分格式(4)是无条件稳定的.

下面进一步讲明格式(3)与格式(4)之间的关系. 若令 $\zeta=4a\Delta t/(\Delta x)^2$, 取近似式 $\lambda=e^{-\zeta}\approx1-\zeta$ 代入格式(4)便得格式(3), 稳定性条件就是 $\zeta\leq1$(即 $r=a\Delta t/(\Delta x)^2\leq\frac{1}{4}$). 这样才能满足 $0\leq\lambda\leq1$. 因此, 有限差分 FTCS 格式相应于近似式 $\lambda=1-\zeta$ 正是它当 $\frac{4a\Delta t}{(\Delta x)^2}>1$ 时引起了数值不稳定, 而这是不必要的. 也许由于 λ 要作指数函数计算不方便, 对此也可作有理式简化. 如 $\lambda=e^{-\zeta}=\frac{1}{e^{\zeta}}\approx(1+\zeta+\frac{\zeta^2}{2})^{-1}$ 等也能满足条件 $0\leq\lambda\leq1$, 从而保证了无条件稳定性, 因此仍比有限差分 FTCS 格式好. 这个性质表明可以采用显式格式, 不同的 FTCS 格式之间仅有的差别只在于参数 λ, 显式格式比隐式格式更方便有效[3]. 因此, 本文只考虑显式格式.

两个 FTCS 格式(3)与(4)皆为 Δt 的一阶精度. 因此还应寻求较好的格式. 蛙跳格式具有二阶精度. 然而, 差分蛙跳格式

$$u^{n+1}_{i,j}=u^{n-1}_{i,j}+\frac{2a\Delta t}{(\Delta x)^2}\{(u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j})+(u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1})\} \tag{5}$$

却是无条件不稳定的. 而单点子域精细积分给出的蛙跳积分格式[2]为

$$u^{n+1}_{i,j}=\left[u^{n-1}_{i,j}-\frac{u^n_{i+1,j}+u^n_{i-1,j}+u^n_{i,j+1}+u^n_{i,j-1}}{4}\right]e^{-\frac{8a\Delta t}{(\Delta x)^2}}+\frac{1}{4}(u^n_{i+1,j}+u^n_{i-1,j}+u^n_{i,j+1}+u^n_{i,j-1}). \tag{6}$$

由于 $0<\lambda^2=e^{-\frac{8a\Delta t}{(\Delta x)^2}}\leq1$, 因此用 Von Neumann 法导出其稳定性条件为 $\frac{1}{2}|(1-\lambda^2)(\cos p\pi\Delta x+\cos q\pi\Delta y)|\leq1-\lambda^2\leq2$ 总是成立的. 所以蛙跳积分格式(6)也是无条件稳定的.

2　蛙跳格式应用于二维对流扩散方程

下面, 应用蛙跳格式于二维对流扩散方程

$$\frac{\partial u}{\partial t}=\epsilon\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right)-a\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right). \tag{7}$$

将它离散化后可表为

$$\frac{\mathrm{d}u_{i,j}}{\mathrm{d}t} = C_1(u_{i-1,j} + u_{i,j-1}) - 2C_2u_{i,j} + C_3(u_{i+1,j} + u_{i,j+1}), \tag{8}$$

其中 C_1, C_2, C_3 为待定参数.

方程(7) 有3个显然的解为

$$u = x + y - 2at, \qquad u = \mathrm{e}^{a(x+y)/\epsilon}, \qquad u = \text{常数}. \tag{9}$$

代入式(8), 可得方程组

$$\left.\begin{aligned} &C_1 + C_3 = C_2, \\ &-C_1(\Delta x_i + \Delta y_j) + C_3(\Delta x_{i+1} + \Delta y_{j+1}) = -2a, \\ &C_1\left(\mathrm{e}^{-\frac{a\Delta x_i}{\epsilon}} + \mathrm{e}^{-\frac{a\Delta y_j}{\epsilon}}\right) - 2C_2 + C_3\left(\mathrm{e}^{\frac{a\Delta x_{i+1}}{\epsilon}} + \mathrm{e}^{\frac{a\Delta y_{j+1}}{\epsilon}}\right) = 0, \end{aligned}\right\} \tag{10}$$

其中 $\Delta x_{i+1} = x_{i+1} - x_i$, $\Delta x_i = x_i - x_{i-1}$, $\Delta y_{j+1} = y_{j+1} - y_j$, $\Delta y_j = y_j - y_{j-1}$. 若令 $p_i = a\Delta x_i/\epsilon$, $p_{i+1} = a\Delta x_{i+1}/\epsilon$, $q_j = a\Delta y_j/\epsilon$, $q_{j+1} = a\Delta y_{j+1}/\epsilon$, 以及

$$F_{i,j} = \frac{2 - \mathrm{e}^{-p_i} - \mathrm{e}^{-q_j}}{\Delta(x_i + y_j)}, \qquad G_{i+1,j+1} = \frac{\mathrm{e}^{p_{i+1}} + \mathrm{e}^{q_{j+1}} - 2}{\Delta(x_{i+1} + y_{j+1})},$$

$$C_d = a\frac{G_{i+1,j+1} + F_{i,j}}{G_{i+1,j+1} - F_{i,j}}$$

则方程组(10) 的解为

$$\left.\begin{aligned} C_1 &= \frac{2a(\mathrm{e}^{p_{i+1}} + \mathrm{e}^{q_{j+1}} - 2)}{\Delta(x_{i+1} + y_{j+1})(\mathrm{e}^{-p_i} + \mathrm{e}^{-q_j} - 2) + \Delta(x_i + y_j)(\mathrm{e}^{p_{i+1}} + \mathrm{e}^{q_{j+1}} - 2)} = \\ &\quad \frac{2aG_{i+1,j+1}}{(G_{i+1,j+1} - F_{i,j})\Delta(x_i + y_j)} = \frac{a + C_d}{\Delta(x_i + y_j)}, \\ C_3 &= \frac{2a(2 - \mathrm{e}^{-p_i} - \mathrm{e}^{-q_j})}{\Delta(x_{i+1} + y_{j+1})(\mathrm{e}^{-p_i} + \mathrm{e}^{-q_j} - 2) + \Delta(x_i + y_j)(\mathrm{e}^{p_{i+1}} + \mathrm{e}^{q_{j+1}} - 2)} = \\ &\quad \frac{2aF_{i,j}}{(G_{i+1,j+1} - F_{i,j})\Delta(x_{i+1} + y_{j+1})} = \frac{-a + C_d}{\Delta(x_{i+1} + y_{j+1})}, \\ C_2 &= C_1 = C_3. \end{aligned}\right\} \tag{11}$$

易证 $C_1 > C_3 > 0$, 因此可说式(8) 是上风格式. 于是, 二维对流扩散方程(7) 的单点子域精细积分蛙跳格式为

$$u_{i,j}^{n+1} = \mathrm{e}^{-4C_2\Delta t}\, u_{i,j}^{n-1} + \frac{(1 - \mathrm{e}^{-4C_2\Delta t})}{2C_2}\{C_1(u_{i-1,j}^n + u_{i,j-1}^n) + C_3(u_{i+1,j}^n + u_{i,j+1}^n)\}. \tag{12}$$

本文结论, 可以容易地推广至更高维情况.

3 稳定性分析

下面应用 Von Neumann 法研究蛙跳积分格式(12) 的稳定性. 令 $\tilde{\lambda} = \mathrm{e}^{-4C_2\Delta t}$, 则 $0 < \tilde{\lambda} < 1$, 且蛙跳格式(12) 成为

$$u_{i,j}^{n+1} = \tilde{\lambda}u_{i,j}^{n-1} + \frac{1 - \tilde{\lambda}}{2C_2}\{C_1(u_{i-1,j}^n + u_{i,j-1}^n) + C_3(u_{i+1,j}^n + u_{i,j+1}^n)\}. \tag{12'}$$

令 $u_{i,j}^n = \rho^n \exp(\mathrm{i}(px_i + qy_j))$ (其中 $\mathrm{i} = \sqrt{-1}$), 代入蛙跳格式(12') 成为

$$\rho = \tilde{\lambda}\rho^{-1} + (1 - \tilde{\lambda})\beta, \tag{13}$$

其中 $\beta = \frac{C_1}{2C_2}(e^{-i(p\Delta x_i)} + e^{-i(q\Delta y_j)}) + \frac{C_3}{2C_2}(e^{i(p\Delta x_{i+1})} + e^{i(q\Delta y_{j+1})})$. 因 $C_1 > C_3 > 0$, $C_2 = C_1 + C_3$, 所以 $\beta \leq \frac{C_1 + C_3}{C_2} = 1$. 改写方程(13)为

$$\rho^2 - (1 - \tilde{\lambda})\beta\rho - \tilde{\lambda} = 0. \tag{13′}$$

由韦达定理, 设 ρ_1, ρ_2 为二次方程(13′)的两根, 则

$$\rho_1\rho_2 = -\tilde{\lambda}, \qquad \rho_1 + \rho_2 = (1 - \tilde{\lambda})\beta. \tag{14}$$

令 $\rho_1 = r_1 e^{i\theta}$, $\rho_2 = -r_2 e^{-i\theta}$, $r_1 > 0$, $r_2 > 0$, 则

$$r_1 r_2 = \tilde{\lambda}, \ (r_1 - r_2)\cos\theta + i(r_1 + r_2)\sin\theta = (1 - \tilde{\lambda})\beta. \tag{15}$$

由式$(15)^2$, 得 $r_1^2 + r_2^2 - 2r_1r_2\cos 2\theta = (1 - \tilde{\lambda})^2\beta^2$, 再与式(15)第 1 式组合得

$$(r_1 + r_2)^2 = (1 - \tilde{\lambda})^2\beta^2 + 2\tilde{\lambda}(1 + \cos 2\theta) \leq$$
$$(1 - \tilde{\lambda})^2 + 4\tilde{\lambda} = (1 + \tilde{\lambda})^2.$$

代入 $r_2 = \tilde{\lambda}/r_1$, 得 $(r_1 + \frac{\tilde{\lambda}}{r_1})^2 \leq (1 + \tilde{\lambda})^2$, 即 $(r_1^2 + \tilde{\lambda})^2 \leq r_1^2(1 + \tilde{\lambda})^2$ 或 $[r_1^2 + \tilde{\lambda} + r_1(1 + \tilde{\lambda})][r_1^2 + \lambda - r_1(1 + \tilde{\lambda})] \leq 0$. 也即 $r_1^2 + \tilde{\lambda} - r_1 - r_1\tilde{\lambda} \leq 0$, 或 $(r_1 - \tilde{\lambda})(r_1 - 1) \leq 0$. 所以, $\tilde{\lambda} \leq r_1 \leq 1$. 同理, 可证 $\lambda \leq r_2 \leq 1$. 因此, 蛙跳积分格式(12)是无条件稳定的. 然而蛙跳差分格式都是数值不稳定的.

4　相容性分析

熟知, 在一般情况下, 不存在无条件相容且无条件稳定的显式差分格式. 因此, 我们必须讨论精细积分格式的相容性, 推导出相容性条件.

(Ⅰ) FTCS 积分格式(4). 改写格式(4)为

$$L_h^{(1)}u_{i,j}^n \equiv \frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + \frac{1}{\Delta t}\left(u_{i,j}^n - \frac{u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n}{4}\right)$$
$$\left(1 - e^{-\frac{4a\Delta t}{(\Delta x)^2}}\right), \tag{4′}$$

并于(x_i, y_j, t_n)处进行 Taylor 展开. 下面为方便起见, 省略上下标 i, j, n 可得

$$L_h^{(1)}u_{i,j}^n = \frac{\partial u}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \cdots - \frac{1}{4\Delta t}\Big\{(\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) +$$
$$\frac{(\Delta x)^4}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + \cdots\Big\}\left(\frac{4a\Delta t}{(\Delta x)^2} - \frac{(4a\Delta t)^2}{2(\Delta x)^4} + \frac{(4a\Delta t)^3}{6(\Delta x)^6} + \cdots\right).$$

所以, FTCS 格式格式(4)的局部截断误差为

$$R_{i,j}^{(1)} = \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{a^2}{6}\Delta t\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) - \frac{a(\Delta x)^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) -$$
$$\frac{2}{9}a^3\left(\frac{\Delta t}{\Delta x}\right)^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) + \frac{2a^2\Delta t}{(\Delta x)^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \cdots =$$
$$O\left(\Delta t + (\Delta x)^2 + \left(\frac{\Delta t}{\Delta x}\right)^2 + \frac{\Delta t}{(\Delta x)^2}\right). \tag{16}$$

(Ⅱ) 蛙跳积分格式(6). 改写格式(6)为

$$L_h^{(2)}u_{i,j}^n \equiv \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} + \frac{1}{2\Delta t}\Big\{u_{i,j}^{n-1} - \frac{1}{4}(u_{i+1,j}^n + u_{i-1,j}^n +$$

$$u_{i,j+1}^n + u_{i,j-1}^n)\} \quad (1 - e^{-\frac{8a\Delta t}{(\Delta x)^2}}), \tag{6'}$$

将它在(x_i, y_j, t_n)处进行Taylor展开,可得

$$L_h^{(2)} u_{i,j}^n \equiv \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} + \cdots + \frac{1}{2\Delta t}\{-\Delta t\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{2}\frac{\partial^2 u}{\partial t^2}) + \cdots -$$

$$\frac{(\Delta x)^2}{4}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - \frac{(\Delta t)^4}{48}(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}) - \cdots\} \quad (\frac{8a\Delta t}{(\Delta x)^2} - \frac{(8a\Delta t)^2}{2(\Delta x)^4} + \frac{(8a\Delta t)^3}{6(\Delta x)^6} + \cdots).$$

所以,蛙跳积分格式(6)的局部截断误差为

$$R_{i,j}^{(2)} = -\frac{a}{12}(\Delta t)^2(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}) + \frac{a^2\Delta t}{3}(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}) + \frac{(\Delta t)^2}{6}(\frac{\partial^3 u}{\partial t^3}) +$$

$$2a(\frac{\Delta t}{\Delta x})^2\frac{\partial^2 u}{\partial t^2} - \frac{4a\Delta t}{(\Delta x)^2}\frac{\partial u}{\partial t} + \frac{a^2}{4}\frac{\Delta t}{(\Delta x)^2}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \cdots =$$

$$O((\Delta t)^2 + (\Delta x)^2 + (\frac{\Delta t}{\Delta x})^2 + \frac{\Delta t}{(\Delta x)^2}). \tag{17}$$

(Ⅲ) 二维对流扩散方程的蛙跳积分格式(12). 为简便起见,对x方向和y方向取等步长,即$\Delta x = \Delta x_{i+1} = \Delta x_i = \Delta y = \Delta y_{j+1} = \Delta y_j$. 于是$p_i = p_{i+1} = \frac{a\Delta x}{\epsilon}$, $\theta_j = \theta_{j+1} = \frac{a\Delta x}{\epsilon}$, 且$F_{i,j} = \frac{1 - e^{-\frac{a\Delta x}{\epsilon}}}{\Delta x}$, $G_{i+1,j+1} = \frac{e^{\frac{a\Delta x}{\epsilon}} - 1}{\Delta x}$, 所以

$$C_d = a\frac{e^{\frac{a\Delta x}{\epsilon}} - e^{-\frac{a\Delta x}{\epsilon}}}{e^{\frac{a\Delta x}{\epsilon}} + e^{-\frac{a\Delta x}{\epsilon}} - 2} = a\frac{\frac{2a\Delta x}{\epsilon} + \frac{1}{3}(\frac{a\Delta x}{\epsilon})^3 + \cdots}{(\frac{a\Delta x}{\epsilon})^2 + \frac{1}{12}(\frac{a\Delta x}{\epsilon})^4 + \cdots}$$

$$= \frac{2\epsilon}{\Delta x}\frac{1 + \frac{1}{6}(\frac{a\Delta x}{\epsilon})^2 + \cdots}{1 + \frac{1}{12}(\frac{a\Delta x}{\epsilon})^2 + \cdots} \approx \frac{2\epsilon}{\Delta x}, C_1 = \frac{a + C_d}{2\Delta x} \approx \frac{a}{2\Delta x} + \frac{\epsilon}{(\Delta x)^2},$$

$$C_3 = \frac{-a + C_d}{2\Delta x} \approx \frac{-a}{2\Delta x} + \frac{\epsilon}{(\Delta x)^2}, C_2 = C_1 + C_3 \approx \frac{2\epsilon}{(\Delta x)^2}.$$

于是,蛙跳积分格式(12)可改写为

$$L_h^{(3)} u_{i,j}^n \equiv \frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} + \frac{1 - e^{-4C_2\Delta t}}{2\Delta t}\{u_{i,j}^{n-1} - \frac{C_1}{2C_2}(u_{i-1,j}^n + u_{i,j-1}^n) -$$

$$\frac{C_3}{2C_2}(u_{i+1,j}^n + u_{i,j+1}^n)\}. \tag{12'}$$

上式于点(x_i, y_j, t_n)处进行Taylor展开,可得

$$L_h^{(3)} u_{i,j}^n = \frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} + \cdots + \frac{4C_2\Delta t - 8C_2^2(\Delta t)^2 + \cdots}{2\Delta t}.$$

$$\{u_{i,j}^n - \Delta t\frac{\partial u}{\partial t} + \frac{1}{2}(\Delta t)^2\frac{\partial^2 u}{\partial t^2} + \cdots - \frac{1}{2C_2}(\frac{a}{2\Delta x} + \frac{\epsilon}{(\Delta x)^2})(u_{i-1,j}^n +$$

$$u_{i,j-1}^n) + \frac{1}{2C_2}(\frac{a}{2\Delta x} - \frac{\epsilon}{(\Delta x)^2})(u_{i+1,j}^n + u_{i,j+1}^n)\} =$$

$$\frac{\partial u}{\partial t} + \frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3} + (\frac{4\epsilon}{(\Delta x)^2} - \frac{16\epsilon^2\Delta t}{(\Delta x)^4})(u_{i,j}^n - \Delta t\frac{\partial u}{\partial t} + \frac{1}{2}(\Delta t)^2\frac{\partial^2 u}{\partial t^2}) +$$

$$(1-4\epsilon\frac{\Delta t}{(\Delta x)^2})\frac{a}{2\Delta x}[(u_{i+1,j}^n-u_{i-1,j}^n)+(u_{i,j+1}^n-u_{i,j-1}^n)]-$$
$$(1-4\epsilon\frac{\Delta t}{(\Delta x)^2})\frac{\epsilon}{(\Delta x)^2}(u_{i+1,j}^n+u_{i-1,j}^n+u_{i,j+1}^n+u_{i,j-1}^n)+\cdots=$$
$$\frac{\partial u}{\partial t}+\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}+(\frac{4\epsilon}{(\Delta x)^2}-\frac{16\epsilon^2\Delta t}{(\Delta x)^4})(u-\Delta t\frac{\partial u}{\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2 u}{\partial t^2})+\cdots)+$$
$$(1-4\epsilon\frac{\Delta t}{(\Delta x)^2})a[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{(\Delta x)^2}{6}(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3})+\cdots]-$$
$$(1-4\epsilon\frac{\Delta t}{(\Delta x)^2})\frac{\epsilon}{(\Delta x)^2}[4u+(\Delta x)^2(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})+\frac{(\Delta x)^4}{12}(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4})+\cdots+]+\cdots$$

于是, 可得蛙跳积分格式(12) 的局部截断误差为

$$\begin{aligned}R_{i,j}^{(3)}&=\frac{a}{6}(\Delta x)^2(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3})-\frac{\epsilon}{12}(\Delta x)^2(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4})-\\&\frac{2}{3}a\epsilon\Delta t(\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3})+\frac{\epsilon^2}{3}\Delta t(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4})+\frac{(\Delta t)^2}{6}\frac{\partial^3 u}{\partial t^3}+\\&2\epsilon(\frac{\Delta t}{\Delta x})^2\frac{\partial^2 u}{\partial t^2}-4\epsilon\frac{\Delta t}{(\Delta x)^2}\frac{\partial u}{\partial t}-4\epsilon a\frac{\Delta t}{(\Delta x)^2}(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y})+\\&4\epsilon^2\frac{\Delta t}{(\Delta x)^2}(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})+\cdots=\\&O(\Delta t+(\Delta x)^2+(\frac{\Delta t}{\Delta x})^2+\frac{\Delta t}{(\Delta x)^2}).\end{aligned}\tag{18}$$

综上所述, 积分格式(3), (4) 和(12) 的局部截断误差阶, 都是 $O(\Delta t+(\Delta x)^2+(\frac{\Delta t}{\Delta x})^2+\frac{\Delta t}{(\Delta x)^2})$ 或 $O(\Delta t)^2+(\Delta x)^2+(\frac{\Delta t}{\Delta x})^2+\frac{\Delta t}{(\Delta x)^2})$. 由此可见, 这些格式的相容性条件为 $\frac{\Delta t}{(\Delta x)^2}\to 0$ (当 $\Delta x\to 0$ 时), 即 $\Delta t=O((\Delta x)^2)$. 这一点在后面的数值例子也已有所反映. 因此, 为使显式格式既相容又无条件稳定, 其代价也是高昂的.

5　数值试验

例 **1**　求解二维扩散方程初边值问题

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}++\frac{\partial^2 u}{\partial y^2}\quad(0<x,y<1,t>0),\\&u(x,y,0)=\sin(x+y)\quad(0\le x,y\le 1),\\&u(0,y,t)=e^{-2t}\sin y,\ u(1,y,t)=e^{-2t}\sin(1+y)\quad(0\le y\le 1,\ t\ge 0),\\&u(x,0,t)=e^{-2t}\sin x,\ u(x,1,t)=e^{-2t}\sin(x+1)\quad(0\le x\le 1,\ t\ge 0).\end{aligned}\right\}\tag{19}$$

其精确解为

$$u(x,y,t)=e^{-2t}\sin(x+y).\tag{20}$$

取 $\Delta x=\Delta y=0.1$, $\Delta t=0.000\ 25, 0.002\ 50$ 及 $0.005\ 00$, 此时 $r=\frac{\Delta t}{(\Delta x)^2}=\frac{\Delta t}{(\Delta y)^2}$ 相应地分别为 0. 025, 0. 250 及 0. 500. 采用单点子域精细积分蛙跳格式(6) 计算到 $n=500$, 并将计算结果与精确解及差分蛙跳格式计算结果列表比较, 如表 1 所示.

表1　蛙跳积分格式(6)计算二维扩散方程初边值问题(19)结果比较表($\Delta x = \Delta y = 0.1, n = 500$)[1]

Δt	(x, y)	精确解式(20)	子域精细积分蛙跳格式(6)	差分蛙跳格式(5)
0.000 25	(0.2, 0.2)	0.303 279 3	0.303 407 6	溢出
	(0.4, 0.4)	0.558 677 5	0.559 002 6	
	(0.6, 0.6)	0.725 872 8	0.726 247 1	
	(0.8, 0.8)	0.778 468 7	0.778 674 0	
0.002 50	(0.2, 0.2)	0.031 965 4	0.033 170 6	溢出
	(0.4, 0.4)	0.058 884 2	0.061 956 9	
	(0.6, 0.6)	0.076 506 4	0.079 977 7	
	(0.8, 0.8)	0.082 050 0	0.083 874 9	
0.005 00	(0.2, 0.2)	0.002 623 9	0.002 997 2	溢出
	(0.4, 0.4)	0.004 833 5	0.005 789 0	
	(0.6, 0.6)	0.006 280 0	0.007 352 0	
	(0.8, 0.8)	0.006 735 1	0.007 287 8	

[1] 蛙跳格式是三层格式,需先用其他方法计算第1层网络函数值. 为方便计,这里采用精确值计算

例2　求解对流扩散方程初边值问题[4]

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= a\frac{\partial u}{\partial x} + \epsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (0 \le x, y \le 2, t > 0), \\ u(x, y, 0) &= \exp\left\{-\frac{(x-0.5)^2 + (y-0.5)^2}{\epsilon}\right\}. \end{aligned}\right\} \tag{21}$$

边界条件由式(27)给出,其精确解为

$$u(x, y, t) = \frac{1}{4t+1}\exp\left\{-\frac{(x-at-0.5)^2 + (y-at-0.5)^2}{\epsilon(4t+1)}\right\}. \tag{22}$$

选取$\epsilon = 0.01$, $a = -0.8$, 当$\tau = 0.012\,5$, $h = 0.1$时计算到$t = 1.25$, 数值结果如表2所示. 又数值解的平均绝对误差为0.006 196 08, 最大绝对误差为0.118 686 58. 数值结果表明, 对流扩散方程的蛙跳积分格式是有效的.

表2　用蛙跳积分格式(12)计算二维对流扩散方程(21)的结果比较表
(取$\epsilon = 0.01$, $a = -0.8$, $h = 0.1$, $\tau = 0.012\,5$计算到$t = 1.25$时的数据)

(x_i, y_i)	精确解式(22)	蛙跳格式式(12)
(0.20, 0.30)	0.000 000 00	0.000 000 00
(0.20, 0.70)	0.000 000 00	0.000 000 00
(0.20, 1.10)	0.000 000 00	0.000 000 00
(0.20, 1.15)	0.000 000 00	0.000 000 00
(0.20, 1.90)	0.000 000 00	0.000 000 00
(0.60, 0.30)	0.000 000 00	0.000 000 00
(0.60, 0.70)	0.000 000 00	0.000 008 02
(0.60, 1.10)	0.000 000 02	0.000 167 70
(0.60, 1.50)	0.000 000 23	0.000 327 51
(0.60, 1.90)	0.000 000 02	0.000 140 51
(1.00, 0.30)	0.000 000 00	0.000 000 07
(1.00, 0.70)	0.000 000 06	0.000 402 02
(1.00, 1.10)	0.000 179 54	0.008 310 56

续表

(x_i, y_i)	精确解式(22)	蛙跳格式式(12)
(1.00, 1.50)	0.002 583 97	0.015 964 09
(1.00, 1.90)	0.000 179 54	0.006 700 40
(1.40, 0.30)	0.000 000 00	0.000 000 21
(1.40, 0.70)	0.000 003 29	0.001 265 21
(1.40, 1.10)	0.09 802 75	0.025 738 78
(1.40, 1.50)	0.141 080 27	0.048 431 59
(1.40, 1.90)	0.009 802 75	0.019 793 74
(1.80, 0.30)	0.000 000 00	0.000 000 12
(1.80, 0.70)	0.000 000 87	0.000 752 78
(1.80, 1.10)	0.002 583 98	0.014 989 27
(1.80, 1.50)	0.037 188 38	0.027 466 67
(1.80, 1.90)	0.002 583 98	0.010 861 11

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A Steady Leapfrog Integration Scheme for Solving Two-Dimensional Convection-Diffusion Equation

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Abstract　For solving two-dimensional convection-diffusion equation, the author gives here an explicit leapfrog integration scheme deriving from single-point precise integration method. The scheme is proved to be unconditionally stable. Its consistency is analysed and the condition of its consisteney is given. The scheme is idicated by numerical example to be effective.

Keywords　two-dimensional convection-diffusion equation, leapfrog integration scheme, unconditionally stable, condition of consistency