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二维对流扩散方程的分步交替分组显式格式

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摘要　将求解二维对流扩散方程的差分方法, 分解成两个一维的情形进行处理, 简化了计算. 该格式还具有绝对稳定性与并行性质, 以及较高的计算精度.

关键词　交替分段显隐方法, 局部一维方法, 分步交替显式格式, 对流扩散方程, 稳定性

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在二维对流扩散方程的数值解中, 显式方法的稳定性条件相当苛刻, 故一般都采用隐式方法. 但是, 隐式方法不具备并行性质, 它一般不适合于并行机或向量机上的计算. 本文先构造出一维对流扩散方程的交替分段显-隐式方法(ASE-I 方法), 再将二维方程进行分解. 尔后, 利用一维的 ASE-I 方法构造出二维的分步交替分组显式格式(AGE 格式). 它既具有明显的并行性质又是绝对稳定的.

1　一维方程的 ASE-I 方法

考虑一维对流扩散方程的初边值问题:

$$\frac{\partial u}{\partial t} + k_1 \frac{\partial u}{\partial x} = d \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le 1,\ t > 0, \tag{1}$$

$$u(x, 0) = f(x), \qquad 0 \le x \le 1, \tag{2}$$

$$u(0, t) = g_1(t),\ u(1, t) = g_2(t), \qquad t > 0. \tag{3}$$

均匀剖分求解区间[0, 1], 其网格点 $x_i = i\Delta x$, $i = 0, 1, \cdots, m$, $\Delta x = 1/m$, $t = k\Delta t$, $k = 0, 1, \cdots$, Δt 为时间步长. 记

$$r = \frac{\Delta t d}{\Delta x^2}, \qquad p = \frac{\Delta t k_1}{2\Delta x},$$

则方程(1)的 [1] 显式格式为

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + k_1 \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = d \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}, \tag{4}$$

$$u_i^{n+1} = (r - p) u_{i+1}^n + (1 - 2r) u_i^n + (r + p) u_{i-1}^n. \tag{5}$$

逼近式(1)的隐式格式为

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$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+k_1\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x}=d\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{\Delta x^2},\tag{6}$$

$$-(r+p)u_{i-1}^{n+1}+(1+2r)u_i^{n+1}-(r-p)u_{i+1}^{n+1}=u_i^n.\tag{7}$$

方程(1)的 Saul´yev 型非对称格式为

$$(1+r)u_i^{n+1}-(r-p)u_{i+1}^{n+1}=(1-r)u_i^n+(r+p)u_{i-1}^n,\tag{8}$$

$$-(r+p)u_{i-1}^{n+1}+(1+r)u_i^{n+1}=(r-p)u_{i+1}^n+(1-r)u_i^n.\tag{9}$$

设 $m-1=kl, k, l$ 均为正整数, $k\geq 3$ 且为奇数, $l\geq 3$. 现将某 $j+1$ 层的 $m-1$ 个内点分为 k 段, 每段包括 l 个点. 由此建立差分格式, 即对某 $i_0\geq 0$, 考虑

$$(i_0+h, k+1)\qquad (h=1,2,\cdots,l)$$

诸点上的计算. 在两个端点分别利用格式(8)与(9), 而在该段的内点$(i_0+h, k+1)$, $(h=2,\cdots,l-1)$ 上使用隐式格式(7). 其描述格式的方程组形式为

$$\begin{bmatrix} 1+r & -(r-p) & & & \\ -(r+p) & 1+2r & -(r-p) & & \\ & \ddots & & & \\ & & -(r+p) & 1+2r & -(r-p) \\ & & & -(r+p) & 1+r \end{bmatrix}\begin{bmatrix} u_{i_0+1}^{n+1} \\ u_{i_0+2}^{n+1} \\ \\ u_{i_0+l-1}^{n+1} \\ u_{i_0+l}^{n+1}\end{bmatrix}=$$

$$\begin{bmatrix}(1-r)u_{i_0+1}^n+(r+p)u_{i_0}^n \\ u_{i_0+2}^n \\ \\ u_{i_0+l-1}^n \\ (1-r)u_{i_0+l}^n+(r-p)u_{i_0+l+1}^n\end{bmatrix}.\tag{10}$$

当 $i_0=0$(即第一段)时, 左端点近旁不用 Saul´yev 型格式, 而取隐式格式(7). 当 $i_0=(k-1)l$(即最后一段)时, 右端边界亦改用隐式格式(7). 令 $q=p/r$, 有

$$(I+r\boldsymbol{G}_1)\boldsymbol{U}^{n+1}=(I-r\boldsymbol{G}_2)\boldsymbol{U}^n+\boldsymbol{b}_1^n,\tag{11}$$

$$(I+r\boldsymbol{G}_2)\boldsymbol{U}^{n+1}=(I-r\boldsymbol{G}_1)\boldsymbol{U}^n+\boldsymbol{b}_2^n,\tag{12}$$

其中 $\boldsymbol{U}^n=(u_1^n,u_2^n,\cdots,u_{m-1}^n)^{\mathrm{T}}$, $u_i^0=f(x_i)(i=0,1,\cdots,m)$, $u_0^n=g_1(t^n)$, $u_m^n=g_2(t^n)(n=1,2,\cdots)$, $\boldsymbol{b}_1^n=(ru_0^n,0,\cdots,0,ru_m^n)^{\mathrm{T}}$, $\boldsymbol{b}_2^n=(ru_0^{n+1},0,\cdots,0,ru_m^{n+1})^{\mathrm{T}}$, $\boldsymbol{G}_1,\boldsymbol{G}_2$ 为 $m\times m$ 矩阵. 定义为

$$\boldsymbol{G}_1=\begin{bmatrix}\boldsymbol{Q}_1 & & & & & & & \\ & \boldsymbol{G}_l^{(1)} & & & & & & \\ & & \boldsymbol{Q}_1 & & & & & \\ & & & \boldsymbol{G}_l^{(2)} & & & & \\ & & & & \ddots & & & \\ & & & & & \boldsymbol{Q}_1 & & \\ & & & & & & \boldsymbol{G}_l^{(\frac{k-1}{2})} & \\ & & & & & & & \boldsymbol{Q}_1\end{bmatrix},\tag{13}$$

$$\boldsymbol{G}_2 = \begin{bmatrix} \hat{\boldsymbol{G}}_{l+1}^{(1)} & & & & & & & \\ & \boldsymbol{Q}_{l-2} & & & & & & \\ & & \boldsymbol{G}_{l+2}^{(2)} & & & & & \\ & & & \boldsymbol{Q}_{l-2} & & & & \\ & & & & \ddots & & & \\ & & & & & \boldsymbol{G}_{l+2}^{(\frac{k-1}{2})} & & \\ & & & & & & \boldsymbol{Q}_{l-2} & \\ & & & & & & & \hat{\boldsymbol{G}}_{l+1}^{(\frac{k+1}{2})} \end{bmatrix}. \tag{14}$$

在式(13),(14)中,有

$$\hat{\boldsymbol{G}}_{l+1}^{(1)} = \begin{bmatrix} 2 & -(1-q) & & & \\ -(1+q) & 2 & -(1-q) & & \\ & & \ddots & & \\ & -(1+q) & 2 & -(1-q) \\ & & -(1+q) & 1 \end{bmatrix}_{(l+1)\times(l+1)}, \tag{15}$$

$$\boldsymbol{G}_{l'}^{(1)} = \begin{bmatrix} 1 & -(1-q) & & & \\ -(1+q) & 2 & -(1-q) & & \\ & & \ddots & & \\ & -(1+q) & 2 & -(1-q) \\ & & -(1+q) & 1 \end{bmatrix}_{(l')\times(l')}, \tag{16}$$

$$\boldsymbol{G}_{l'}^{(2)} = \cdots = \boldsymbol{G}_{l'}^{(\frac{k-1}{2})} = \boldsymbol{G}_{l'}^{(1)}, \qquad l' = l \text{ 或 } l+2,$$

$$\hat{\boldsymbol{G}}_{l+1}^{(\frac{k+1}{2})} = \begin{bmatrix} 1 & -(1-q) & & & \\ -(1+q) & 2 & -(1-q) & & \\ & & \ddots & & \\ & -(1+q) & 2 & -(1-q) \\ & & -(1+q) & 2 \end{bmatrix}_{(l+1)\times(l+1)}, \tag{17}$$

在式(15)～(17)中,$l' = l$,或 $l' = l-2$,而 $\boldsymbol{Q}_l$ 为 $l'\times l$ 零阵.

2 二维对流扩散方程的分步 AGE 方法

考虑二维对流扩散方程的初边值问题[]:

$$\frac{\partial u}{\partial t} + k_1\frac{\partial u}{\partial x} + k_2\frac{\partial u}{\partial y} = d_1\frac{\partial^2 u}{\partial x^2} + d_2\frac{\partial^2 u}{\partial y^2},$$

$$0 < x, \quad y < 1, \quad 0 < t \leq T, \tag{18}$$

$$u(x, y, 0) = \varphi(x, y), \quad 0 < x, y < 1, \tag{19}$$

$$u(0, y, t) = f_0(y, t), \quad u(1, y, t) = f_1(y, t), \quad 0 < y < 1,\ 0 < t \leq T, \tag{20}$$

$$u(x, 0, t) = g_0(x, t), \quad u(x, 1, t) = g_1(x, t), \quad 0 < x < 1,\ 0 < t \leq T. \tag{21}$$

下面,用分步格式来构造二维对流扩散问题 AGE 方法. 为方便起见,记

$$\Lambda_x^{1,\theta}u_{ij}^n=\begin{cases}d_1(u_{i+1,j}^{n+\theta}-u_{i,j}^{n+\theta}-u_{i,j}^n+u_{i-1,j}^n)/\Delta x^2-k_1(u_{i+1,j}^{n+\theta}-u_{i-1,j}^n)/2\Delta x,\\ d_1(u_{i+1,j}^{n+\theta}-2u_{i,j}^{n+\theta}+u_{i-1,j}^{n+\theta})/\Delta x^2-k_1(u_{i+1,j}^{n+\theta}-u_{i-1,j}^{n+\theta})/2\Delta x,\\ d_1(u_{i+1,j}^n-2u_{i,j}^n+u_{i-1,j}^n)/\Delta x^2-k_1(u_{i+1,j}^n-u_{i-1,j}^n)/2\Delta x,\end{cases}\tag{22}$$

$$\Lambda_x^{2,\theta}u_{ij}^n=\begin{cases}d_1(u_{i+1,j}^n-u_{i,j}^n-u_{i,j}^{n+\theta}+u_{i-1,j}^{n+\theta})/\Delta x^2-k_1(u_{i+1,j}^n-u_{i-1,j}^{n+\theta})/2\Delta x,\\ d_1(u_{i+1,j}^{n+\theta}-2u_{i,j}^{n+\theta}+u_{i-1,j}^{n+\theta})/\Delta x^2-k_1(u_{i+1,j}^{n+\theta}-u_{i-1,j}^{n+\theta})/2\Delta x,\\ d_1(u_{i+1,j}^n-2u_{i,j}^n+u_{i-1,j}^n)/\Delta x^2-k_1(u_{i+1,j}^n-u_{i-1,j}^n)/2\Delta x,\end{cases}\tag{23}$$

$$\Lambda_y^{1,\theta}u_{ij}^n=\begin{cases}d_2(u_{i,j+1}^{n+\theta}-u_{i,j}^{n+\theta}-u_{i,j}^n+u_{i,j-1}^n)/\Delta y^2-k_2(u_{i,j+1}^{n+\theta}-u_{i,j-1}^n)/2\Delta y,\\ d_2(u_{i,j+1}^{n+\theta}-2u_{i,j}^{n+\theta}+u_{i,j-1}^{n+\theta})/\Delta y^2-k_2(u_{i,j+1}^{n+\theta}-u_{i,j-1}^{n+\theta})/2\Delta y,\\ d_2(u_{i,j+1}^n-2u_{i,j}^n+u_{i,j-1}^n)/\Delta y^2-k_2(u_{i,j+1}^n-u_{i,j-1}^n)/2\Delta y,\end{cases}\tag{24}$$

$$\Lambda_y^{2,\theta}u_{ij}^n=\begin{cases}d_2(u_{i,j+1}^n-u_{i,j}^n-u_{i,j}^{n+\theta}+u_{i,j-1}^{n+\theta})/\Delta y^2-k_2(u_{i,j+1}^n-u_{i,j-1}^{n+\theta})/2\Delta y,\\ d_2(u_{i,j+1}^{n+\theta}-2u_{i,j}^{n+\theta}+u_{i,j-1}^{n+\theta})/\Delta y^2-k_2(u_{i,j+1}^{n+\theta}-u_{i,j-1}^{n+\theta})/2\Delta y,\\ d_2(u_{i,j+1}^n-2u_{i,j}^n+u_{i,j-1}^n)/\Delta y^2-k_2(u_{i,j+1}^n-u_{i,j-1}^n)/2\Delta y.\end{cases}\tag{25}$$

由于偏微分方程不含混合导数项，所以可以采用简单分步方法．即

$$\Lambda_t^{\frac{1}{2}}u_{ij}^n=\Lambda_x^{s,\frac{1}{2}}u_{ij}^n,\quad s=1,2,\tag{26}$$

$$\Lambda_t^{\frac{1}{2}}u_{ij}^{n+\frac{1}{2}}=\Lambda_y^{l,\frac{1}{2}}u_{ij}^{n+\frac{1}{2}},\quad l=1,2,\tag{27}$$

其中

$$\Lambda_t^{\theta}u_{ij}^n=\frac{1}{\Delta t}(u_{i,j}^{n+\theta}-u_{i,j}^n).\tag{28}$$

也即分解成两个一维情形来处理．为方便起见，假定 $\Delta x=\Delta y=h,x_i=i\Delta x,y_j=j\Delta y,i,j=0,1,\cdots,m$．并设 m 为偶数．记 $r_1=\frac{\Delta td_1}{(\Delta x)^2},p_1=\frac{\Delta tk_1}{2\Delta x},r_2=\frac{\Delta td_2}{(\Delta y)^2},p_2=\frac{\Delta tk_2}{2\Delta y},q_1=\frac{p_1}{r_1},q_2=\frac{p_2}{r_2}$．由一维情形的交替分段显-隐式方法，可构造出二维问题的分步 AGE 方法：

$$(I+r_1C_1)U^{n+\frac{1}{2}}=(I-r_1C_2)U^n+B,\tag{29}$$

$$(I+r_2\hat{C}_1)\hat{U}^{n+1}=(I-r_2\hat{C}_2)\hat{U}^{n+\frac{1}{2}}+\hat{B},\tag{30}$$

$$(I+r_1C_2)U^{n+\frac{3}{2}}=(I-r_1C_1)U^{n+1}+D,\tag{31}$$

$$(I+r_2\hat{C}_2)\hat{U}^{n+2}=(I-r_2\hat{C}_1)\hat{U}^{n+\frac{3}{2}}+\hat{D},\tag{32}$$

其中 $\boldsymbol{U}^n=(u_{1,1}^n,u_{2,1}^n,\cdots,u_{m,1}^n,\cdots,u_{1,m}^n,u_{2,m}^n,\cdots,u_{m,m}^n)^{\mathrm{T}}$，$\hat{\boldsymbol{U}}^n=(u_{1,1}^n,u_{1,2}^n,\cdots,u_{1,m}^n,\cdots,u_{m,1}^n,u_{m,2}^n,\cdots,u_{m,m}^n)^{\mathrm{T}}$．并有

$$\boldsymbol{C}_1=\begin{bmatrix}\boldsymbol{G}_1^{(1)}&&&\\&\boldsymbol{G}_1^{(2)}&&\\&&\ddots&\\&&&\boldsymbol{G}_1^{(m)}\end{bmatrix},\quad \boldsymbol{C}_2=\begin{bmatrix}\boldsymbol{G}_2^{(1)}&&&\\&\boldsymbol{G}_2^{(2)}&&\\&&\ddots&\\&&&\boldsymbol{G}_2^{(m)}\end{bmatrix},\tag{33}$$

其中 $\boldsymbol{G}_1^{(1)}=\boldsymbol{G}_1^{(2)}=\cdots=\boldsymbol{G}_1^{(m)}=\boldsymbol{G}_1,\boldsymbol{G}_2^{(1)}=\boldsymbol{G}_2^{(2)}=\cdots=\boldsymbol{G}_2^{(m)}=\boldsymbol{G}_2$，$\boldsymbol{G}_1,\boldsymbol{G}_2$ 分别如式(15)，(16)所定义，只是 $r=r_1,p=p_1,q=q_1$．

$$\hat{\boldsymbol{C}}_1=\begin{bmatrix}\boldsymbol{G}_1^{(1)}&&&\\&\hat{\boldsymbol{G}}_1^{(2)}&&\\&&\ddots&\\&&&\hat{\boldsymbol{G}}_1^{(m)}\end{bmatrix},\quad \hat{\boldsymbol{C}}_2=\begin{bmatrix}\hat{\boldsymbol{G}}_2^{(1)}&&&\\&\hat{\boldsymbol{G}}_2^{(2)}&&\\&&\ddots&\\&&&\hat{\boldsymbol{G}}_2^{(m)}\end{bmatrix},\tag{34}$$

其中 $\hat{G}_1^{(1)}=\hat{G}_1^{(2)}=\cdots=\hat{G}_1^{(m)}=\hat{G}_1$, $\hat{G}_2^{(1)}=\hat{G}_2^{(2)}=\cdots=\hat{G}_2^{(m)}=\hat{G}_2$. 而 $\hat{G}_1$, $\hat{G}_2$ 分别类似 G_1, G_2, 只是 $r=r_2$, $p=p_2$, $q=q_2$. $B=(b_i)$, $b_j=(r_1u_{0,j}^n, 0, \cdots, 0, r_1u_{m,j}^n)^T$, $\hat{B}=(\hat{b}_i)$, $\hat{b}_i=(r_2u_{i,0}^{n+\frac{1}{2}}, 0, \cdots, 0, r_2u_{i,m}^{n+\frac{1}{2}})^T$, $D=(d_j)$, $d_j=(r_1u_{0,j}^{n+\frac{3}{2}}, 0, \cdots, 0, r_1u_{m,j}^{n+\frac{3}{2}})^T$, $\hat{D}=(\hat{d}_i)$, $\hat{d}_i=(r_2u_{i,0}^{n+2}, 0, \cdots, 0, r_2u_{i,m}^{n+2})^T$, $(i, j=1, 2, \cdots, m)$.

3 稳定性分析

3.1 一维 ASE-I 方法的稳定性

先给出两个引理[2].

引理 1(Kellog 引理) 设 $\rho>0$. 如果 $B+B^T$ 为非负定矩阵, 那么有估计式 $\|(I-\rho B)(I+\rho B)^{-1}\|_2\leq 1$.

引理 2 $G_1+G_1^T$, $G_2+G_2^T$, 是非负定矩阵(证略).

由此易得

定理 1 ASE-I 方法是绝对稳定的.

3.2 二维方法的稳定性

在稳定性讨论中, 假定边界处理是精确的. 那么, 对于讨论稳定性时, AGE 格式可表示为

$$(I+r_1C_1)U^{n+\frac{1}{2}}=(I-r_1C_2)U^n, \tag{35}$$

$$(I+r_1\hat{C}_1)\hat{U}^{n+1}=(I-r_2\hat{C}_2)\hat{U}^{n+\frac{1}{2}}, \tag{36}$$

$$(I+r_1C_2)U^{n+\frac{3}{2}}=(I-r_1C_1)U^{n+1}, \tag{37}$$

$$(I+r_2\hat{C}_2)\hat{U}^{n+2}=(I-r_2\hat{C}_1)\hat{U}^{n+\frac{3}{2}}. \tag{38}$$

定义

$$\bar{U}_n=\begin{bmatrix} u_{1,1}^n & u_{1,2}^n & \cdots & u_{1,m}^n \\ u_{2,1}^n & u_{2,2}^n & \cdots & u_{2,m}^n \\ & & & \\ u_{m,1}^n & u_{m,2}^n & \cdots & u_{m,m}^n \end{bmatrix}.$$

将式(35)～(38)分别改写为

$$(I+r_1G_1)U^{n+\frac{1}{2}}=(I-r_1G_2)U^n, \tag{39}$$

$$(I+r_2\hat{G}_1)(\bar{U}^{n+1})^T=(I-r_2\hat{G}_2)(\bar{U}^{n+\frac{1}{2}})^T, \tag{40}$$

$$(I+r_1G_2)\bar{U}^{n+\frac{3}{2}}=(I-r_1G_1)\bar{U}^{n+1}, \tag{41}$$

$$(I+r_2\hat{G}_2)(\bar{U}^{n+2})^T=(I-r_2\hat{G}_1)(\bar{U}^{n+\frac{3}{2}})^T. \tag{42}$$

消去两个中间层 $t=n+\dfrac{1}{2}$, $n+\dfrac{3}{2}$, 得

$$\bar{U}^{n+1}=P_1\bar{U}^nH_1, \qquad \bar{U}^{n+2}=P_2\bar{U}^{n+1}H_2, \tag{43}$$

其中 $P_1=(I+r_1G_1)^{-1}(I-r_1G_2)$, $P_2=(I+r_1G_2)^{-1}(I-r_1G_1)$, $H_1=(I-r_2\hat{G}_2^T)(I+r_2\hat{G}_1^T)^{-1}$, $H_2=(I-r_2\hat{G}_1^T)(I+r_2\hat{G}_2^T)^{-1}$. 进一步得

$$\bar{U}^{n+2}=P_2P_1\bar{U}^nH_1H_2. \tag{44}$$

记 $\boldsymbol{P}=\boldsymbol{P}_1\boldsymbol{P}_2$, $\boldsymbol{H}=\boldsymbol{H}_1\boldsymbol{H}_2=(h_{ji})_{m\times m}$, 取 $\boldsymbol{T}_1=\mathrm{diag}(\boldsymbol{P}^{(1)},\boldsymbol{P}^{(2)},\cdots,\boldsymbol{P}^{(m)})$, $\boldsymbol{P}^{(1)}=\boldsymbol{P}^{(1)}=\cdots=\boldsymbol{P}^{(m)}=\boldsymbol{P}$, $\boldsymbol{T}_2=(\boldsymbol{T}_{ij})$, $\boldsymbol{T}_{ij}=h_{ij}\boldsymbol{I}_m$. 另记 $\boldsymbol{U}_j^n=(u_{1,j}^n,u_{2,j}^n,\cdots,u_{m,j}^n)^{\mathrm{T}}$, $j=1,2,\cdots,m$, 则

$$\boldsymbol{U}^n=(\boldsymbol{U}_1^n,\boldsymbol{U}_2^n,\cdots,\boldsymbol{U}_m^n),\qquad \boldsymbol{U}^n=[\boldsymbol{U}_1^n,\boldsymbol{U}_2^n,\cdots,\boldsymbol{U}_m^n]^{\mathrm{T}}.$$

将式(44)改写为

$$(\boldsymbol{U}_1^{n+2},\boldsymbol{U}_2^{n+2},\cdots,\boldsymbol{U}_m^{n+2})=\boldsymbol{P}(\boldsymbol{U}_1^n,\boldsymbol{U}_2^n,\cdots,\boldsymbol{U}_m^n)\boldsymbol{H}. \tag{45}$$

由式(45)有

$$\begin{aligned}\boldsymbol{U}_j^{n+2}&=\boldsymbol{P}\boldsymbol{U}_1^n h_{1j}+\boldsymbol{P}\boldsymbol{U}_2^n h_{2j}+\cdots+\boldsymbol{P}\boldsymbol{U}_m^n h_{mj}=h_{1j}\boldsymbol{P}\boldsymbol{U}_1^n h_{2j}\boldsymbol{P}\boldsymbol{U}_2^n+\cdots+h_{mj}\boldsymbol{P}\boldsymbol{U}_m^n=\\ &\boldsymbol{T}_{j1}\boldsymbol{P}\boldsymbol{U}_1^n+\boldsymbol{T}_{2j}\boldsymbol{P}\boldsymbol{U}_2^n+\cdots+\boldsymbol{T}_{jm}\boldsymbol{P}\boldsymbol{U}_m^n,\ j=1,2,\cdots,m.\end{aligned} \tag{46}$$

故

$$\boldsymbol{U}^{n+2}=\begin{bmatrix}\boldsymbol{U}_1^{n+2}\\ \boldsymbol{U}_2^{n+2}\\ \\ \boldsymbol{U}_m^{n+2}\end{bmatrix}=\begin{bmatrix}\boldsymbol{T}_{11}\boldsymbol{P} & \boldsymbol{T}_{12}\boldsymbol{P} & \cdots & \boldsymbol{T}_{1m}\boldsymbol{P}\\ \boldsymbol{T}_{21}\boldsymbol{P} & \boldsymbol{T}_{22}\boldsymbol{P} & \cdots & \boldsymbol{T}_{2m}\boldsymbol{P}\\ \\ \boldsymbol{T}_{m1}\boldsymbol{P} & \boldsymbol{T}_{m2}\boldsymbol{P} & \cdots & \boldsymbol{T}_{mm}\boldsymbol{P}\end{bmatrix}\begin{bmatrix}\boldsymbol{U}_1^n\\ \boldsymbol{U}_2^n\\ \\ \boldsymbol{U}_m^n\end{bmatrix}=$$

$$\begin{bmatrix}\boldsymbol{T}_{11} & \boldsymbol{T}_{12} & \cdots & \boldsymbol{T}_{1m}\\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} & \cdots & \boldsymbol{T}_{2m}\\ \\ \boldsymbol{T}_{m1} & \boldsymbol{T}_{m1} & \cdots & \boldsymbol{T}_{mm}\end{bmatrix}\begin{bmatrix}\boldsymbol{P} & & & \\ & \boldsymbol{P} & & \\ & & \ddots & \\ & & & \boldsymbol{P}\end{bmatrix}\begin{bmatrix}\boldsymbol{U}_1^n\\ \boldsymbol{U}_2^n\\ \\ \boldsymbol{U}_m^n\end{bmatrix}=\boldsymbol{T}_2\boldsymbol{T}_1\boldsymbol{U}^n. \tag{47}$$

即

$$\boldsymbol{U}^{n+2}=\boldsymbol{T}_2\boldsymbol{T}_1\boldsymbol{U}^n. \tag{48}$$

由定理 1 知, $\|\boldsymbol{P}\|_2=\|\boldsymbol{P}_2\boldsymbol{P}_1\|_2=\|(\boldsymbol{I}+r_2\boldsymbol{G}_2)^{-1}(\boldsymbol{I}-r_1\boldsymbol{G}_1)(\boldsymbol{I}+r_1\boldsymbol{G}_1)^{-1}(\boldsymbol{I}-r_2\boldsymbol{G}_2)\|_2\le 1$. 同理, 易得 $\|\boldsymbol{H}\|_2\le 1$. 再由 $\|\boldsymbol{T}_1\|_2=\|\boldsymbol{P}\|_2$, $\|\boldsymbol{T}_2\|_2=\|\boldsymbol{H}\|_2$, 有

$$\|\boldsymbol{T}_1\boldsymbol{T}_2\|_2\le\|\boldsymbol{T}_1\|_2\|\boldsymbol{T}_2\|_2\le 1.$$

这便证明了 AGE 方法是绝对稳定的. 因此有定理 2.

定理 2　AGE 方法是绝对稳定的.

4　数值试验与结论

方程 $\dfrac{\partial u}{\partial t}+k_1\dfrac{\partial u}{\partial x}+k_2\dfrac{\partial u}{\partial y}=d_1\dfrac{\partial^2 u}{\partial x^2}+d_2\dfrac{\partial^2 u}{\partial y^2}$, $0<x,y<2$, $0<t\le T$, 在 $\Omega=\{(x,y)\ 0\le x,y\le 2\}$ 区域上. 给定初始条件

$$u(x,y,0)=\exp\left\{-\frac{(x-0.5)^2}{d_1}-\frac{(y-0.5)^2}{d_2}\right\}$$

的解析解为[3]

$$u(x,y,t)=\frac{1}{4t+1}\exp\left\{-\frac{(x-k_1t-0.5)^2}{d_1(4t+1)}-\frac{(y-k_2t-0.5)^2}{d_2(4t+1)}\right\}. \tag{49}$$

边界条件由式(49)给出.

选取 $k_1=k_2=0.8$, $d_1=d_2=0.01$. 当 $\tau=0.0125$, $h=0.1$ 时, AGE 方法的数值解的平均绝对误差为 0.0062, 最大绝对误差为 0.09654, 最小绝对误差为 0. 其数值解与解析解的三维比较图, 如图 1 所示. 从绝对误差与图形的拟合情况来看, AGE 方法计算结果较为理想, 其计算

精度是高的. 由于是显式计算, 因此它又具有很好的并行性.

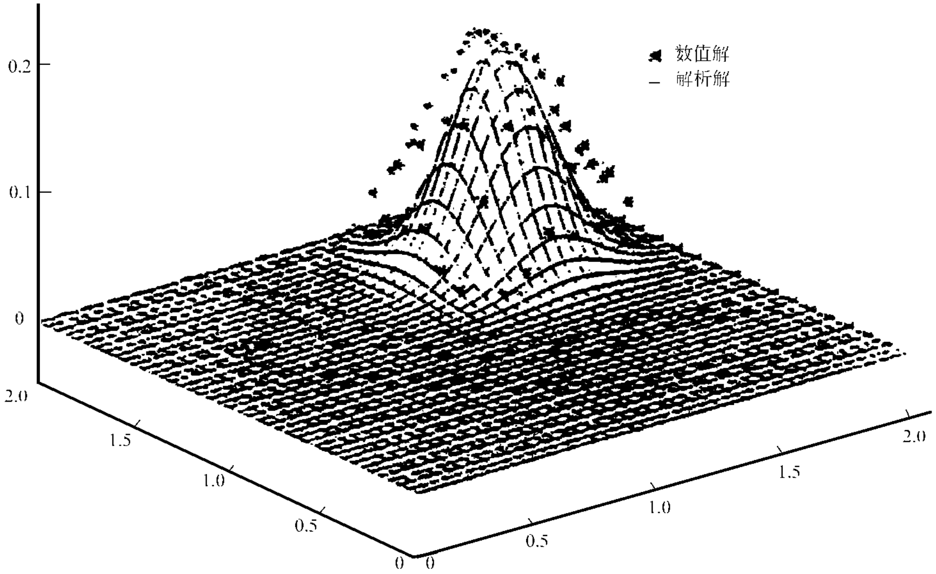

图1 数值结果比较图

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Fractional and Alternate and Grouping and Explicit Schemes for Solving Two-Dimensional Convection-Diffusion Equation

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Abstract The difference method for solving two-dimensional convection-diffusion equation is decomposed into two one-dimensional circumstances by which the calculation is simplified. The scheme shows absolutely stable and parallel character, it is high precision in calculation.

Keywords alternately piecewise explicit-implicit method, local one-dimensional method, fractional and alternate and explicit scheme, convection-diffusion equation, stability