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两端弹性铰支约束下压杆稳定的优化设计

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摘要 采用泛函极值分析的拉格朗日乘子法, 对压杆在两端弹性铰支约束下进行截面优化的稳定分析. 求得压杆在任意弹性铰支约束下, 截面优化分布规律和挠曲函数的通用表达式, 并给出计算临界荷载的统一公式.

关键词 压杆, 边界条件, 优化设计, 临界荷载

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压杆稳定的截面优化问题, 较早就为人们所重视. 1960 年, Keller 对两端铰支压杆进行稳定分析^[1], 得到了截面优化分布规律, 及其相应的失稳挠曲函数和临界荷载. 1962 年, Tadjbakhsh 和 Keller^[2]又解决了两端固支, 一端固支和另端铰支, 以及一端固支和另端自由等特殊边界条件约束下, 压杆截面优化问题^[2]. 但这些边界约束均是理想条件下的约束, 实际的边界约束应是介于铰支与固支(自由端除外)的约束, 即弹性铰支约束. 将压杆边界约束考虑为具有广泛意义的弹性铰支约束, 求解出该边界条件约束下压杆的截面优化分布规律和失稳挠曲函数. 同时, 给出便于理论计算和实际应用的临界荷载统一公式, 并对所得结果的适用性进行讨论.

1 问题的提出

设计一个两端弹性铰支约束的压杆(图 1), 杆长为 l , 杆的体积为 V , 所用材料的弹性模量为 E , 两端弹簧扭转刚度分别为 α_a 和 α_b . 其截面惯性矩与截面面积的关系为 $I(x) = kA^2(x)$. 确定压杆面积的分布, 使压杆临界荷载 P_{cr} 达到最大, 其数学表达式为

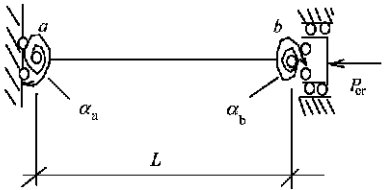


图 1 两端弹性铰支约束压杆

$$\left. \begin{aligned} \Pi &= \int_0^l \left[\frac{1}{2} k E A^2(x) y'^2(x) - \frac{1}{2} P y'^2(x) \right] dx \\ y(0) &= y(l) = 0, \quad V = \int_0^l A(x) dx, \\ \alpha_a y'(0) &= M_a, \quad \alpha_b y'(l) = -M_b. \end{aligned} \right\} \quad (1)$$

2 问题的求解

从式(1)可知,此问题为带定积分约束条件的泛函极值问题.应用拉格朗日乘子法,可将条件极值方便地化为无条件极值问题^[6].因此,构造一个新泛函为

$$\Pi^* = \int_0^l F dx = \int_0^l \left[\frac{1}{2} kEA^2(x) y'^2(x) - \frac{1}{2} P y'^2 - \lambda \right] dx, \quad (2)$$

式中 λ 为拉格朗日乘子.新泛函驻值条件为

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) - \frac{d^2}{dx^2} \left(\frac{\partial F}{\partial y} \right) = 0, \quad \frac{\partial F}{\partial \lambda} = 0. \quad (3)$$

注意边界条件式(1)中第2式,由式(3)得

$$kEA^2 y' + P y = M_a + (M_b - M_a) \cdot \frac{x}{l}, \quad kEA y'^2 - \lambda = 0. \quad (4)$$

整理式(4),加上边界条件,得定解问题为

$$\left. \begin{aligned} y &= \left(\frac{\lambda^2}{kEP} \right)^{1/3} \left[-y + \frac{M_a}{P} + \frac{(M_b - M_a)x}{P} \cdot \frac{x}{l} \right]^{-1/3}, \\ A &= \left(\frac{P^2}{kE\lambda} \right)^{1/3} \left[-y + \frac{M_a}{P} + \frac{(M_b - M_a)x}{P} \cdot \frac{x}{l} \right]^{2/3}, \\ y(0) &= y(l) = 0, \quad \alpha y(0) = M_a, \quad \alpha y(l) = -M_b. \end{aligned} \right\} \quad (5)$$

作变量代换 $\xi = -y + \frac{M_a}{P} + \frac{(M_b - M_a)x}{P} \cdot \frac{x}{l}$, 即

$$y = -\xi + \frac{M_a}{P} + \frac{(M_b - M_a)x}{P} \cdot \frac{x}{l}. \quad (6)$$

将式(6)代入式(5),得

$$\left. \begin{aligned} \xi &= - \left(\frac{\lambda^2}{kEP} \right)^{1/3} \xi^{-1/3}, \quad A = \left(\frac{P^2}{kE\lambda} \right)^{1/3} \xi^{2/3}, \quad \xi(0) = \frac{M_a}{P}, \quad \xi(l) = \frac{M_b}{P} \\ \xi(0) &= \frac{[\xi(l) - \xi(0)]}{l} - \frac{P}{\alpha} \xi(0), \quad \xi(l) = \frac{[\xi(l) - \xi(0)]}{l} + \frac{P}{\alpha} \xi(l). \end{aligned} \right\} \quad (7)$$

由 $\xi = \frac{d\xi}{dx} = \frac{d\xi}{d\xi} \frac{d\xi}{dx} = \xi \frac{d\xi}{d\xi}$, 对式(7)第1式积分,得

$$\xi^2 = C - 3 \left(\frac{\lambda^2}{kEP} \right)^{1/3} \xi^{2/3}. \quad (8)$$

将 $\xi(x)$ 表示的挠曲函数沿杆长分两段分析,应有

$$\left. \begin{aligned} \xi &= - \frac{3}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{1/6} \frac{1 - (\xi/\xi_c)^{2/3}}{1 - (\xi_c/\xi_c)^{2/3}} \quad (0 \leq x \leq l_c), \\ \xi_a &= \frac{M_a}{P}, \quad \xi_a = \frac{(\xi_b - \xi_a)}{l} - \beta \xi_a, \quad \xi(l_c) = \xi_c, \quad \xi(l_c) = 0, \\ \xi &= - \frac{3}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{1/6} \frac{1 - (\xi/\xi_c)^{2/3}}{1 - (\xi/\xi_c)^{2/3}} \quad (l_c \leq x \leq l), \\ \xi(l_c) &= \xi_c, \quad \xi(l_c) = 0, \quad \xi_b = \frac{M_a}{P}, \quad \xi_b = \frac{(\xi_b - \xi_a)}{l} + \beta \xi_b, \end{aligned} \right\} \quad (9)$$

其中 $\xi(0) = \xi_a$, $\xi(l) = \xi_b$, $P/\alpha = \beta$, $P/\alpha = \beta$. 作变量代换 $(\xi/\xi_c)^{1/3} = \sin \theta$ 即

将式(10)代入式(9)积分(注意边界条件), 有

$$\left. \begin{aligned} 2\theta - \sin 2\theta &= -\frac{4}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{-2/3} x + 2\theta_a - \sin 2\theta_a, & 0 &\leq x \leq l_c, \\ 2\theta - \sin 2\theta &= -\frac{4}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{-2/3} (l - x) + 2\theta_b - \sin 2\theta_b, & l_c \leq x \leq l. \end{aligned} \right\} \quad (11)$$

并且, 有

$$\left. \begin{aligned} \xi_a &= -\frac{1}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{1/3} \cos \theta_a = \xi_c \left[\frac{\sin^3 \theta_b - \sin^3 \theta_a}{l} - \beta_a \sin^3 \theta_a \right], \\ \xi_b &= -\frac{1}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{1/3} \cos \theta_b = \xi_c \left[\frac{\sin^3 \theta_b - \sin^3 \theta_a}{l} + \beta_b \sin^3 \theta_b \right], \end{aligned} \right\} \quad (12)$$

式中 $\theta_a = \theta(0)$, $\theta_b = \theta(l)$. 将 $x = l_c$, $\sin \theta(l_c) = 1$, $\theta(l_c) = \pi/2$ 代入式(11), 两式相加得

$$2(\theta_a - \theta_b) - (\sin 2\theta_a + \sin 2\theta_b) + \frac{4}{3} \left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{-2/3} l = 2\pi \quad (13)$$

整理式(12)和式(13), 得

$$\left(\frac{\lambda^2}{kEP} \right)^{1/6} \xi_c^{-2/3} = -\frac{3}{4l} [2\pi - 2(\theta_a + \theta_b) + (\sin 2\theta_a + \sin 2\theta_b)]. \quad (14)$$

$$\left. \begin{aligned} (\sin^3 \theta_b - \sin^3 \theta_a)(\cos \theta_a + \cos \theta_b) + \beta_b l \sin^3 \theta_b \cos \theta_b - \beta_a l \sin^3 \theta_a \cos \theta_a &= 0, \\ l(\beta_b \sin^3 \theta_b + \sin^3 \theta_a) + \frac{3}{4} [2\pi - 2(\theta_a + \theta_b) + (\sin 2\theta_a + \sin 2\theta_b)] &= 0, \\ (\sin 2\theta_a + \sin 2\theta_b)(\cos \theta_a + \cos \theta_b) &= 0. \end{aligned} \right\} \quad (15)$$

θ_a 和 θ_b 由式(5)计算出. 将式(10)代入式(7)中第2式得

$$A = \left(\frac{P^2}{kE\lambda} \right)^{1/3} \xi^{2/3} \sin^2 \theta \quad (16)$$

将式(11)和式(16)代入式(1)中第2式, 积分得

$$\begin{aligned} V &= \int_0^l A dx = \int_0^{l_c} A dx + \int_{l_c}^l A dx = -\frac{3}{4} \left(\frac{P^5}{kE\lambda^4} \right)^{1/6} \xi_c^{4/3} \cdot \\ &\quad \left[-\frac{3}{2}(\theta_a + \theta_b) + (\sin 2\theta_a + \sin 2\theta_b) - \frac{1}{8}(\sin 4\theta_a + \sin 4\theta_b) \right], \end{aligned} \quad (17)$$

将式(14)代入式(17), 且把 P 记为 P_{cr} , 得

$$P_{cr} = \frac{4kE}{3l^4} V^2 \frac{[\pi - (\theta_a + \theta_b) + \frac{1}{2}(\sin 2\theta_a + \sin 2\theta_b)]^4}{[\pi - (\theta_a + \theta_b) + \frac{2}{3}(\sin 2\theta_a + \sin 2\theta_b) - \frac{1}{12}(\sin 4\theta_a + \sin 4\theta_b)]^2}. \quad (18)$$

记

$$\mu^2 = \frac{\left[1 - \frac{(\theta_a + \theta_b)}{\pi} + \frac{2}{3\pi}(\sin 2\theta_a + \sin 2\theta_b) - \frac{1}{12\pi}(\sin 4\theta_a + \sin 4\theta_b) \right]^2}{\left[1 - \frac{(\theta_a + \theta_b)}{\pi} + \frac{1}{2\pi}(\sin 2\theta_a + \sin 2\theta_b) \right]^4}, \quad (19)$$

$$P_{\alpha} = \frac{4}{3} \frac{\pi^2 k E}{l^4} \left(\frac{V^2}{\mu^2} \right). \quad (20)$$

由式(20)可知,与等直杆临界荷载相似,若将 V/l 作为同体积等直杆的横截面积,式(20)可改写为

$$P_{\alpha} = \frac{4}{3} \cdot \frac{\pi^2 k E}{3(\mu l)^2} \left(\frac{V^2}{l} \right) = \frac{\pi^2 E I_e}{3(\mu l)^2}, \quad (21)$$

式中 I_e 为等直杆截面惯性矩. 将式(14), (18)代入式(16), 得压杆截面分布规律为

$$A = \frac{4\rho V}{3l} \sin^2 \theta. \quad (22)$$

而

$$\rho = \frac{\left[1 - \frac{(\theta_a + \theta_b)}{\pi} + \frac{1}{2\pi}(\sin 2\theta_a + \sin 2\theta_b) \right]}{1 - (\sin^3 \theta_b - \sin^3 \theta_a) + \frac{2}{3\pi}(\sin 2\theta_a + \sin 2\theta_b) - \frac{1}{12\pi}(\sin 4\theta_a + \sin 4\theta_b)}. \quad (23)$$

将式(12)代入式(6), 得压杆失稳挠曲函数为

$$y = -\xi_c [(\sin^3 \theta - \sin^3 \theta_a) + x/l(\sin^3 \theta_b - \sin^3 \theta_a)], \quad (24)$$

式中 $\theta(x)$ 由式(11)计算得. 式(11)中的 l_c 表示为

$$l_c = \frac{(\pi - 2\theta_a + \sin 2\theta_a)l}{2\pi - 2(\theta_a + \theta_b) + (\sin 2\theta_a + \sin 2\theta_b)}. \quad (25)$$

值得注意的是, 式(18)因 θ_a 和 θ_b 包含 β_a, β_b , 亦即包含 P_{α} 的超越方程. 除特殊约束外, 一般先假定 β_a, β_b , 由式(15)求解 θ_a 和 θ_b . 再由式(18)确定 P_{α} 值, 进而由 $\alpha_a = P_{cr}/\beta_a, \alpha_b = P_{cr}/\beta_b$ 逆解对应的弹簧扭转刚度.

3 结果与讨论

本文的压杆优化稳定分析, 适用于任意两端弹性铰支约束条件. 当弹簧扭转刚度 $\alpha = 0$, 所在端则为铰支约束; 而当 $\alpha = \infty$, 所在端即是固支端. 两端铰支约束, $\mu = 1$; 两端固支约束, $\mu = 0.5$; 一端固支, 另一端铰支, 则 $\mu = 0.7$. 计算临界荷载时系数 μ 之值, 与文献 [1, 2] 完全相同.

由分析可知, 与等直杆相应, μ 也可称为压杆长度系数, μl 为优化杆的计算长度. 应注意在典型约束条件下, 优化杆与等直杆的长度系数 μ 相同, 并不表示所有的约束条件下, 其 μ 值相同. 例如, 当 $\beta_a = \beta_b = 10/L$ 时, 由式(20)所得 $\mu = 0.754$, 而等直杆的 $\mu = 0.81^{[6]}$.

上述分析从实际角度来看, 存在一个问题, 这就是优化杆中有 $A = 0$ 的截面出现. Olhoff 和 Rasmussen^[6]曾指出, 文 [1] 中两端固支压杆的解答不正确. 认为固支优化杆可能相应于两种独立无关的失稳型式, 并求出该问题的数值解, 其临界荷载为同体积等直杆的 1.326 2 倍. 其实在以上分析中, 压杆材料应处于线弹性范围内. 在 $A = 0$ 截面附近的杆段上, 显然材料因受压进入了塑性状态, 不满足材料为线弹性的要求. 故在优化杆设计中, 应在 $A = 0$ 处附近适当加大截面, 以满足理论假设和实际要求^[6]. 现以两端铰支压杆为例, 说明加粗处理措施.

两端铰支压杆的临界荷载, 由式(20)得

$$P_{\alpha} = \frac{4}{3} \cdot \frac{\pi^2 k E}{l^4} V^2. \quad (26)$$

由式(22)和(11)得压杆截面优化分布(另一半对称)为

$$A = \frac{4}{3} \cdot \frac{\pi^2 k E}{l^4} V^2, \quad 2\theta - \sin 2\theta - \frac{2\pi}{l} x = 0, \quad 0 \leq x \leq l/2. \quad (27)$$

加大截面处所需截面面积, 则

$$A_0 = P_{cr} / \sigma_p, \quad (28)$$

式中 σ_p 为材料比例极限. 令 $A_0 = A(\theta_0) = P_{cr}$, 由式(26), (27) 得

$$\left. \begin{aligned} \sin^2 \theta_0 &= \frac{\pi^2 k E V}{l^3 \sigma_p}, \\ 2\theta_0 - \sin^2 \theta_0 - \frac{2\pi}{l} x_0 &= 0. \end{aligned} \right\} \quad (29)$$

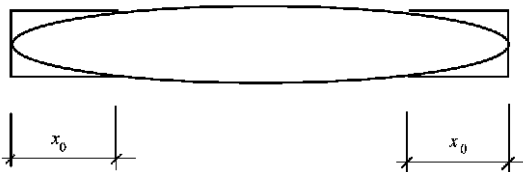


图 2 细处截面加大示意图

式(29)即为加大段位置, 如图 2 所示. 从实际考虑, 将压杆细处加大, 则式(20)计算的临界荷载为优化压杆的荷载上限.

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Optimal Design of the Stability of Compression Bar under the Constraint of Elastically Hinged Support at Both Ends

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Abstract By adopting Lagrange's multipliers method for functional analysis of extreme value, the author makes an optimal stability analysis of a section of compression bar under the constraint of elastically hinged support at both ends. A common expression is obtained for expressing the rule of optimal distribution on the section and the function of flexure of the compression bar under the constraint of arbitrarily elastically hinged support. And an integrated formula is given for calculating critical load.

Keywords compression bar, boundary condition, optimal design, critical load