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解二维抛物型方程的恒稳高精度格式*

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摘要 建立了解二维抛物型方程的一族含参数的绝对稳定的高精度的差分格式,进而,在特殊情况($\theta=0, r=\frac{1}{6}$)下,得到显式差分格式 $\omega^{n+1}=(1+\frac{1}{36}\quad+\frac{1}{9}\diamond)\omega^n$. 这些格式对任意选取的参数 $\theta\leq 1/6$ 都是绝对稳定的,且当 $0\leq\theta\leq\min(\frac{1}{6},\frac{1}{2}-\frac{1}{12r})$ 时,其收敛阶为 $O((\Delta t)^2)$.

关键词 二维抛物型方程,差分格式,高精度,绝对稳定

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渗流、扩散、热体导等很多领域,经常会遇到求解如下的二维抛物型方程的初边值问题

$$\left.\begin{array}{ll}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} & (0\leq x,y\leq l, t>0),\\ u(0,y,t)=g_1(y,t), u(l,y,t)=g_2(y,t) & (0\leq y\leq l, t>0),\\ u(x,0,t)=h_1(x,t), u(x,l,t)=h_2(x,t) & (0\leq x\leq l, t>0),\\ u(x,y,0)=\varphi(x,y) & (0\leq x,y\leq l).\end{array}\right\} \tag{1}$$

解上述问题的差分格式[1,2]大部分是隐式格式且精度不高,其截断误差阶仅为 $O(\Delta t+(\Delta x)^2)$ 或 $O((\Delta t)^2+(\Delta x)^2)$. 文[3,4]进一步提出截断误差阶为 $O((\Delta t)^2+(\Delta x)^4)$ 的高精度隐式或显式差分格式. 本文构造了一族两层含参数、绝对稳定、高精度的隐式差分格式(特殊情况下是显式),包含了文[3]中的高精度恒稳格式. 当参数 $\theta=0$, 网格比 $r=\frac{\Delta t}{\Delta x^2}=\frac{\Delta t}{\Delta y^2}=\frac{1}{6}$ 时,是一个两层的显式差分格式. 可以证明,本文构造的一族含参数高精度格式当参数 $\theta\leq\frac{1}{6}$ 时是恒稳的,且为 $0\leq\theta\leq\min(\frac{1}{2}-\frac{1}{12r},\frac{1}{6})$ 时,其收敛阶为 $O((\Delta t)^2)$.

1 高精度差分格式的构造

设 Δt 为时间步长, $\Delta x,\Delta y$ 分别为 x, y 方向的空间步长,且为简便计,设 $\Delta x=\Delta y=\frac{l}{N}$($N$ 为正整数). 用如下含参数的差分方程逼近微分方程(1),得

$$\frac{\omega_{i,j}^{n+1}-\omega_{i,j}^{n}}{\Delta t}=\frac{1}{(\Delta x)^2}\{\frac{\theta_1}{2}\quad+\theta_2\diamond\}\omega_{i,j}^{n+1}+\frac{1}{(\Delta x)^2}\{\frac{\theta_3}{2}\quad+\theta_4\diamond\}\omega_{i,j}^{n}, \tag{2}$$

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其中 $\omega_{i,j}^n$ 表示在节点$(i\Delta x, j\Delta y, n\Delta t)$处的网格函数, 且记

$$
\left.\begin{aligned}
\Box\omega_{i,j}^n &= \omega_{i+1,j+1}^n + \omega_{i-1,j+1}^n + \omega_{i+1,j-1}^n + \omega_{i-1,j-1}^n - 4\omega_{i,j}^n,\\
\Diamond\omega_{i,j}^n &= \omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n - 4\omega_{i,j}^n.
\end{aligned}\right\} \tag{3}
$$

今后为简便计, 略去下标 i, j 而记 $\omega_{i,j}^n = \omega^n$, 其余类推. $\theta_1, \theta_2, \theta_3, \theta_4$ 为待定参数, 适当选取这些参数, 可以使差分格式(2)逼近微分方程(1)具有尽可能高阶的离散误差, 而且有较好的稳定性.

当微分方程(1)的解充分光滑时, 有如下关系式

$$
\frac{\partial^n}{\partial t^n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^m \omega = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{m+2n}\omega \tag{4}
$$

成立. 由此在节点$(i\Delta x, j\Delta y, (n+\frac{1}{2})\Delta t)$处进行 Taylor 展开, 得

$$
\begin{aligned}
\frac{1}{2(\Delta x)^2}\Box\omega^{n+1} &= \frac{\partial\omega^{n+\frac{1}{2}}}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2\omega^{n+\frac{1}{2}}}{\partial t^2} + \frac{(\Delta x)^2}{12}\left[\frac{\partial^4}{\partial x^4} + 6\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right]\omega^{n+\frac{1}{2}} + \\
&\quad O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4),\\
\frac{1}{2(\Delta x)^2}\Box\omega^{n} &= \frac{\partial\omega^{n+\frac{1}{2}}}{\partial t} - \frac{\Delta t}{2}\frac{\partial^2\omega^{n+\frac{1}{2}}}{\partial t^2} + \frac{(\Delta x)^2}{12}\left[\frac{\partial^4}{\partial x^4} + 6\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right]\omega^{n+\frac{1}{2}} + \\
&\quad O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4),\\
\frac{1}{(\Delta x)^2}\Diamond\omega^{n+1} &= \frac{\partial\omega^{n+\frac{1}{2}}}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2\omega^{n+\frac{1}{2}}}{\partial t^2} + \frac{(\Delta x)^2}{12}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right)\omega^{n+\frac{1}{2}} + \\
&\quad O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4),\\
\frac{1}{(\Delta x)^2}\Diamond\omega^{n} &= \frac{\partial\omega^{n+\frac{1}{2}}}{\partial t} - \frac{\Delta t}{2}\frac{\partial^2\omega^{n+\frac{1}{2}}}{\partial t^2} + \frac{(\Delta x)^2}{12}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right)\omega^{n+\frac{1}{2}} + \\
&\quad O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4),\\
\frac{\omega^{n+1} - \omega^n}{\Delta t} &= \frac{\partial\omega^{n+\frac{1}{2}}}{\partial t} + O((\Delta t)^2).
\end{aligned}
$$

将上述各式代入差分方程(2), 并利用关系式(4), 可得

$$
\begin{aligned}
&\frac{\omega^{n+1} - \omega^n}{\Delta t} - \left\{\frac{1}{(\Delta x)^2}\left[\frac{\theta_1}{2}\Box + \theta_2\Diamond\right]\omega^{n+1} + \frac{1}{(\Delta x)^2}\left[\frac{\theta_3}{2}\Box + \theta_4\Diamond\right]\omega^n\right\} = \\
&\{1 - (\theta_1 + \theta_2 + \theta_3 + \theta_4)\}\frac{\partial\omega^{n+\frac{1}{2}}}{\partial t} - (\theta_1 + \theta_2 - \theta_3 - \theta_4)\frac{\Delta t}{2}\frac{\partial^2\omega^{n+\frac{1}{2}}}{\partial t^2} - \\
&\frac{1}{12}(\theta_1 + \theta_2 + \theta_3 + \theta_4)(\Delta x)^2\frac{\partial^2\omega^{n+\frac{1}{2}}}{\partial t^2} + \frac{(\Delta x)^4}{6}[2(\theta_1 + \theta_3) - (\theta_2 + \theta_4)]\frac{\partial^4\omega^{n+\frac{1}{2}}}{\partial x^2\partial y^2} + \\
&O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4).
\end{aligned} \tag{5}
$$

于是, 我们得到差分格式(2)的截断误差阶为

(Ⅰ) 当满足相容性条件

$$
\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1 \tag{6}
$$

时, 截断误差阶为 $O(\Delta t + (\Delta x)^2)$.

(Ⅱ) 当满足条件

$$
\theta_1 + \theta_2 = \theta_3 + \theta_4 = \frac{1}{2} \tag{7}
$$

时, 截断误差阶为 $O((\Delta t)^2 + (\Delta x)^2)$.

(Ⅲ) 当满足条件

$$\theta_1 + \theta_3 = \frac{1}{3}, \qquad \theta_2 + \theta_4 = \frac{2}{3}, \qquad \theta_1 + \theta_2 = \frac{1}{2} - \frac{1}{12r} \tag{8}$$

时, 截断误差阶为 $O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4) = O((\Delta t)^2)$.

丢掉截断误差后, 便得形如式(2)的差分格式. 下面我们指出一些熟知的格式及本文所得的新的差分格式.

(Ⅰ) 截断误差阶为 $O(\Delta t + (\Delta x)^2)$ 的格式, 即参数满足相容性条件的差分格式(2). 特别地, 有: (A) $\theta_1 = \theta_2 = \theta_3 = 0, \theta_4 = 1$, 即古典显式格式; (B) $\theta_1 = \theta_3 = \theta_4 = 0, \theta_2 = 1$, 即古典隐式格式; (C) $\theta_1 = 1, \theta_2 = \theta_3 = \theta_4 = 0$, 即文[2]格式(Ⅰ); (D) $\theta_1 = \frac{1}{3}, \theta_2 = \frac{2}{3}, \theta_3 = \theta_4 = 0$, 即文[2]格式(Ⅱ). 此类格式尚可列出很多, 从略.

(Ⅱ) 截断误差阶为 $O((\Delta t)^2 + (\Delta x)^2)$ 的差分格式. 若令 $\theta_1 = \theta$, $\theta_3 = \eta$, 则由条件(7)得

$$\theta_1 = \theta, \theta_2 = \frac{1}{2} - \theta, \quad \theta_3 = \eta, \theta_4 = \frac{1}{2} - \eta. \tag{9}$$

於是, 得截断误差阶为 $O((\Delta t)^2 + (\Delta x)^2)$ 的一般格式为

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{1}{2(\Delta x)^2}\{\theta \quad + (1 - 2\theta)\diamond\}\omega^{n+1} + \frac{1}{2(\Delta x)^2}\{\eta \quad + (1 - 2\eta)\diamond\}\omega^n. \tag{10}$$

特别地, 有: (A) $\theta_1 = \theta_3 = \frac{1}{2}, \theta_2 = \theta_4 = 0$, 即文[2]格式(Ⅲ); (B) $\theta_1 = \theta_3 = \frac{1}{6}, \theta_2 = \theta_4 = \frac{1}{3}$, 即文[2]格式(Ⅳ); (C) $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{1}{4}$, 即得新格式为

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{1}{8(\Delta x)^2}(\quad + 2\diamond)(\omega^{n+1} + \omega^n). \tag{11}$$

(Ⅲ) 截断误差阶为 $O((\Delta t)^2 + (\Delta x)^4)$ 的格式. 若令 $\theta_1 = \theta$, 则由条件(8)得

$$\theta_1 = \theta, \quad \theta_2 = \frac{1}{2} - \frac{1}{12r} - \theta, \quad \theta_3 = \frac{1}{3} - \theta, \quad \theta_4 = \frac{1}{6} + \frac{1}{12r} + \theta. \tag{12}$$

于是, 截断误差阶为 $O((\Delta t)^2 + \Delta t(\Delta x)^2 + (\Delta x)^4)$ 的一般格式为

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{1}{2(\Delta x)^2}\{[\theta \quad + (1 - \frac{1}{6r} - 2\theta)\diamond]\omega^{n+1} + [(\frac{1}{3} - \theta) \quad + (\frac{1}{3} + \frac{1}{6r} + 2\theta)\diamond]\omega^n\}. \tag{13}$$

特别地, 有: (A) $\theta = 0$ 为熟知的格式, 即

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{1}{(\Delta x)^2}(\frac{1}{2} - \frac{1}{12r})\diamond\omega^{n+1} + \frac{1}{(\Delta x)^2}[\frac{1}{6} \quad + (\frac{1}{6} + \frac{1}{12r})\diamond]\omega^n; \tag{14}$$

(B) $\theta = \frac{1}{6}$, 此为文[3]中所构造的高精度格式, 即

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{1}{(\Delta x)^2}\{[\frac{1}{12} \quad + (\frac{1}{3} - \frac{1}{12r})\diamond]\omega^{n+1} + [\frac{1}{12} \quad + (\frac{1}{3} + \frac{1}{12r})\diamond]\omega^n\}; \tag{15}$$

(C) $\theta = \frac{1}{6} - \frac{1}{12r}$, 即得新的高精度格式为

$$\frac{\omega^{n+1} - \omega^n}{\Delta t} = \frac{1}{(\Delta x)^2}\{[(\frac{1}{12} - \frac{1}{24r}) \quad + \frac{1}{3}\diamond]\omega^{n+1} + [(\frac{1}{12} + \frac{1}{24r}) \quad + \frac{1}{3}\diamond]\omega^n\}; \tag{16}$$

(D) $\theta = 0, r = \frac{1}{6}$时, 得新的两层的显式格式为

$$\omega^{n+1} = \left(1 + \frac{1}{36} \quad + \frac{1}{9}\Diamond\right)\omega^{n}. \tag{17}$$

由于具有一个自由参数, 我们还可以列出很多高精度格式, 如取 $\theta = -2r\left(\frac{1}{2} - \frac{1}{12r}\right)$ 或 $\theta = -2r\left(\frac{1}{2} - \frac{1}{12r}\right)^2$ 等等, 这些格式也是无条件稳定的.

2　差分格式的稳定性分析

假定边界条件和右端的计算是精确的, 而在初始层产生误差 ϵ, 易知误差

$$v_{i,j}^{n} = W_{i,j}^{n} - \widetilde{W}_{i,j}^{n}. \tag{18}$$

(其中 $W_{i,j}^{n}$ 为差分方程的近似解), 满足如下方程和条件:

$$\left.\begin{aligned} &\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} = \frac{1}{(\Delta x)^2}\left\{\frac{\theta_1}{2} \quad + \theta_2\Diamond\right\}v_{i,j}^{n+1} + \\ &\qquad \frac{1}{(\Delta x)^2}\left\{\frac{\theta_3}{2} \quad + \theta_4\Diamond\right\}v_{i,j}^{n} \quad (i,j = 1, 2\cdots, N-1), \\ &v_{i,j}^{0} = \epsilon \quad (i,j = 0, 1, \cdots, N), \\ &v_{i,j}^{n} = 0 \quad (i, j = 0 \text{ 或 } N). \end{aligned}\right\} \tag{19}$$

令 $v_{i,j}^{n} = \rho_{pq}^{n}\sin(p\pi x_i)\sin(q\pi y_j)$, 记 $A = s_1^2 + s_2^2 = \sin^2\left(\frac{p\pi}{2N}\right) + \sin^2\left(\frac{q\pi}{2N}\right)$, $B = 2(s_1^2 + s_2^2 - 2s_1^2 s_2^2) = 1 - \cos\left(\frac{p\pi}{N}\right)\cos\left(\frac{q\pi}{N}\right) = \sin^2\frac{p+q}{2N}\pi + \sin^2\frac{p-q}{2N}\pi$, $s_1 = \sin\frac{p\pi}{2N}$, $s_2 = \sin\frac{q\pi}{2N}$. 于是, $\Diamond(\sin(p\pi x_i)\sin(p\pi y_i)) = -4A\sin(p\pi x_i)\sin(q\pi y_j)$, $(\sin(p\pi x_i)\sin(q\pi y_j)) = -4B\sin(p\pi x_i)\sin(q\pi y_j)$.

基本定理 **1**　若参数 $\theta_1, \theta_2, \theta_3, \theta_4$ 满足下列条件之一: (i) $\theta_1 + \theta_3 \geq 0$, $\theta_2 + \theta_4 \geq 0$ 且 $\theta_3 \leq \theta_1$, $\theta_4 \leq \theta_2$; (ii) $\theta_1 + \theta_2 = \theta_3 + \theta_4 = \frac{1}{2}$, $\theta_1 + \theta_3 \leq 1$ 且 $\theta_1 \leq \theta_3$; (iii) 对任意 $s_1^2, s_2^2 \in [0, 1]$ 及 $r > 0$ 均成立: (1) $(\theta_1 + \theta_2 + \theta_3 + \theta_4)(s_1^2 + s_2^2) - 2(\theta_1 + \theta_3)s_1^2 s_2^2 \geq 0$; (2) $2[(\theta_3 + \theta_4) - (\theta_1 + \theta_2)]r(s_1^2 + s_2^2) + 4(\theta_1 - \theta_3)rs_1^2 s_2^2 \leq 1$. 因此, 差分格式(2) 无条件稳定.

证明　根据分离变量法可得差分格式(2) 的传播矩阵为

$$G = \frac{1 - 2\theta_3 rB - 4\theta_4 rA}{1 + 2\theta_1 rB + 4\theta_2 rA}. \tag{20}$$

若对任意的 $r > 0$ 和 $s_1^2, s_2^2 \in [0, 1]$ 都成立 $G \leq$, 则差分格式(2) 无条件稳定.

由 S_1, S_2 及 A, B 表达可知, $0 \leq s_1^2 \leq 1$, $0 \leq s_2^2 \leq 1$ 且 $0 \leq A \leq 2$, $0 \leq B \leq 2$, 则

$$G \leq 1 \Leftrightarrow$$

$$\left.\begin{aligned} &2(\theta_1 + \theta_3)rB + 4(\theta_2 + \theta_4)rA \geq 0, \\ &(\theta_3 - \theta_1)rB + 2(\theta_4 - \theta_2)rA \leq 1, \end{aligned}\right\} \Leftrightarrow \tag{21}$$

$$\left.\begin{aligned} &(\theta_1 + \theta_2 + \theta_3 + \theta_4)(s_1^2 + s_2^2) - 2(\theta_1 + \theta_3)s_1^2 s_2^2 \geq 0, \\ &2[(\theta_3 + \theta_4) - (\theta_1 + \theta_2)]r(s_1^2 + s_2^2) + 4(\theta_1 - \theta_3)rs_1^2 s_2^2 \leq 1. \end{aligned}\right\} \tag{22}$$

由式(21) 成立得充分条件(i), 式(22) 即充分条件(iii). 显然, 当充分条件(ii) 成立时式(22) 也成立, 从而完成了基本定理的证明. 由此, 易得如下推论.

推论 1 差分格式(Ⅰ)的(B),(C),(D)格式绝对稳定.

推论 2 差分格式(10)绝对稳定的一个充分条件是 $\theta\le\eta$ 且 $\theta+\eta\le 1$. 由此立得

推论 3 差分格式(Ⅱ)的(A),(B),(C)格式绝对稳定.

推论 4 高精度差分格式(13)绝对稳定的充分条件是 $\theta\le\frac{1}{6}$. 于是立刻有

推论 5 高精度差分格式(14)～(16)是绝对稳定的.

推论 6 高精度两层显格式(17)(此时 $r=\frac{1}{6}$)是稳定的.

3 高精度格式的收敛性

现在讨论高精度一般格式(13)的收性.

引理 1 假设:(i) $0\le\theta\le\frac{1}{2}-\frac{1}{12r}$;(ii) $\frac{1}{2(\Delta x)^2}\{(1-\frac{1}{6r}-2\theta)\diamond+\theta\}Z_{i,j}-\rho Z_{i,j}=g_{i,j}$, 其中 $i,j=1,2,\cdots,N-1$, ρ 为正的常数;(iii) 在边界上 $Z_{i,j}=0(i,j=0$ 或 $N)$. 因此, 有 $\max\limits_{0\le i,j\le N} Z_{i,j}\le\max\limits_{0\le i,j\le N}\left|\frac{g_{i,j}}{\rho}\right|$. 证明时, 可仿照文[3]用反证法证明之. 从略.

基本定理 2 若在区域 $\mathbf{R}=\{0\le x,y\le l,0\le t\le T\}$ 上边值问题(1)的解 U 存在且充分光滑, 则当 $0\le\theta\le\min\{\frac{1}{2}-\frac{1}{12r},\frac{1}{6}\}$ 时高精度一般格式(13)的解 W 收敛于 U, 且其收敛阶为 $O((\Delta t)^2)$.

证明 令 $Z_{i,j,n}=U^n_{i,j}-W^n_{i,j}$, 则误差 $Z_{i,j,n}$ 满足以下方程和条件:

$$\frac{1}{2(\Delta x)^2}\{(1-\frac{1}{6r}-2\theta)\diamond+\theta\}Z_{i,j,n+1}+\frac{1}{2(\Delta x)^2}\{(\frac{1}{3}+\frac{1}{6r}+2\theta)\diamond+(\frac{1}{3}-\theta)\}Z_{i,j,n}=\frac{Z_{i,j,n+1}-Z_{i,j,n}}{\Delta t}+g_{i,j,n}\quad(i,j=1,2,\cdots,N-1),\tag{23}$$

$$Z_{i,j,0}=0\qquad(i,j=0,1,\cdots,N),\tag{24}$$

$$Z_{i,j,n}=0\qquad(i,j=0\text{或}N),\tag{25}$$

其中 $g_{i,j,n}=O((\Delta t)^2)$. 为简便计, 今后略去下标 i,j. 令

$$Z_n=\sum_{k=0}^{n-1}Z_n^{(k)},\tag{26}$$

代入方程(23)可得

$$\frac{1}{2(\Delta x)^2}\{[(1-\frac{1}{6r}-2\theta)\diamond+\theta]\sum_{k=0}^{n}Z_{n+1}^{(k)}\}+\frac{1}{2(\Delta x)^2}\{[(\frac{1}{3}+\frac{1}{6r}+2\theta)\diamond+(\frac{1}{3}-\theta)]\sum_{k=0}^{n-1}Z_n^{(k)}\}=\{\sum_{k=0}^{n}Z_{n+1}^{(k)}-\sum_{k=0}^{n-1}Z_n^{(k)}\}/\Delta t+g_n.$$

当 $k=0,1,2,\cdots,n-1$ 时, $Z_n^{(k)}$ 和 $Z_{n+1}^{(k)}$ 满足下列方程, 即

$$\frac{1}{2(\Delta x)^2}\{(1-\frac{1}{6r}-2\theta)\diamond+\theta\}Z_{n+1}^{(k)}+\frac{1}{2(\Delta x)^2}\{(\frac{1}{3}+\frac{1}{6r}+2\theta)\diamond+(\frac{1}{3}-\theta)\}Z_n^{(k)}=\frac{1}{\Delta t}\{Z_{n+1}^{(k)}-Z_n^{(k)}\}.$$

当 $k=n$ 时, 有

$$\frac{1}{2(\Delta x)^2}\{(1-\frac{1}{6r}-2\theta)\Diamond Z_{k+1}^{(k)}+\theta\quad Z_{k+1}^{(k)}\}=\frac{Z_{k+1}^{(k)}}{\Delta t}+g_k.$$

同时, 将式(26)代入齐次边界条件(24), (25), 得

$$Z_0=\sum_{k=0}^{n-1}Z_0^{(k)}=0,\qquad Z_n=\sum_{k=0}^{n-1}Z_n^{(k)}=0.$$

故有 $Z_0^{(k)}=0$ 及 $Z_n^{(k)}=0$ $(k=0,1,2,\cdots,n-1)$. 根据定理的假设条件及上述两式, 并应用引理 1, 可得

$$\max_{i,j}\ Z_{k+1}^{(k)}\ \leq\Delta t\ \ \max\ g_k\ \ \leq 0((\Delta t)^3).$$

假若将 $Z_{k+1}^{(k)}$, 展成有限的 Fourier 级数 $Z_{n+1}^{(k)}=\sum_{p,q=1}^{N-1}C_{pq}^{(k)}\rho_{pq}^{n-k}\sin p\pi x\ \sin q\pi y$, 那么有 $Z_n^{(k)}=\sum_{p,q=1}^{N-1}C_{pq}^{(k)}\rho_{pq}^{n-k-1}\sin p\pi x\sin q\pi y$, $Z_n=\sum_{k=0}^{n-1}\sum_{p,q=1}^{N-1}C_{pq}^{(k)}\rho_{pq}^{n-k-1}\sin p\pi x\sin q\pi y$, 其中

$$\rho_{pq}=\frac{1-\frac{1}{3}A-4(\frac{1}{6}+\theta)rA-2r(\frac{1}{3}-\theta)B}{1-\frac{1}{3}A+4(\frac{1}{2}-\theta)rA+2r\theta B}=$$

$$\frac{1-\frac{1}{3}(s_1^2+s_2^2)-2r\ [(s_1^2+s_2^2)-4(\frac{1}{3}-\theta)s_1^2s_2^2]}{1-\frac{1}{3}(s_1^2+s_2^2)+2r\ [(s_1^2+s_2^2)-4\theta s_1^2s_2^2]}.\tag{27}$$

由 Fourier 系数的引理[], 可知

$$C_{pq}^{(k)}=O((\frac{1}{pq}+\frac{1}{N^{2\epsilon}})(\Delta t)^3)\qquad 1\leq p,q\leq N^{1-\epsilon},$$

$$C_{pq}^{(k)}=O((\Delta t)^3)\qquad N^{1-\epsilon}<p\text{ 或 }q<N-1.$$

当 $\theta\leq\frac{1}{2}$时, $s_1^2+s_2^2-4\theta s_1^2s_2^2\geq s_1^2+s_2^2-2s_1^2s_2^2\geq 0$, 故当 $1\leq p,q\leq 1$ 时, 式(27)的分母始终为正. 下面, 分两种情况估计 ρ_{pq}.

(i) 当式(27)分子为正时, 注意到当 $\theta\geq 0$ 时有 $s_1^2+s_2^2-2(1-3\theta)s_1^2s_2^2\geq 0$, 从而我们有

$$\rho_{pq}<1-\frac{2}{3}r(s_1^2+s_2^2).\tag{28}$$

若令 $\sigma_1=p\pi/2N$, $\sigma_2=p\pi/2N$, 则当 $1\leq p,q\leq N-1$ 时, $0<\sigma_1,\sigma_2<\pi/2$. 对于给定的 $r>0$, 只要 σ_1,σ_2 足够小, 总能使式(27)的分子具有正值. 即存在 $\sigma_0>0$ 使得当 N 充分大且 p,q 适当小, 以致 $0<\sigma_1,\sigma_2<\sigma_0$ 时, 式(27)的分子为正. 此时估计式(28)成立.

(ii) 当 $\sigma_0\leq\sigma_1,\sigma_2<\pi/2$ 时, 若式(27)的分子为负, 则当 $\theta\leq\frac{1}{6}$时, 有

$$\rho_{pq}<1-\frac{1-\frac{1}{3}(s_1^2+s_2^2)}{1-\frac{1}{3}(s_1^2+s_2^2)+2r(s_1^2+s_2^2)-8r\theta s_1^2s_2^2}.$$

综上所述可知, 当 $0<\sigma_1,\sigma_2<\pi/2$ 时, 总有 $\rho_{pq}<1-\eta$, 其中

$$\eta=\min\{\frac{2}{3}r(s_1^2+s_2^2),\frac{1-\frac{1}{3}(s_1^2+s_2^2)}{1-\frac{1}{3}(s_1^2+s_2^2)+2r(s_1^2+s_2^2)-8r\theta s_1^2s_2^2}\}.$$

从而对一切 $1 \le p, q \le N-1$, 总有

$$\rho_{pq} < \begin{cases} 1 - C_1 N^{-2} & \text{当 } 1 \le p, q \le N^{1-\epsilon}, \\ 1 - C_2 N^{-2\epsilon} & \text{当 } p \text{ 或 } q > N^{1-\epsilon}. \end{cases}$$

进而由文 [] 可知 $Z_n = O(N^{-4}(\lg N)^2 + N^{-2-4\epsilon} + N^{-4+2\epsilon})$. 由于当 $N \longrightarrow +\infty$ 时首项为高阶无穷小量, 故可选取 ϵ, 使得 $-2-4\epsilon = -4+2\epsilon$. 即选取 $\epsilon = \frac{1}{3}$, 从而有 $Z_n = O((\Delta t)^{5/3})$, 仿照文 [] 和文 [3], 将 p, q 进一步细分可得估计式 $Z_n = O((\Delta t)^2)$, 从略. 基本定理 2 证毕.

推论 1 高精度差分格式(14) ~(17), 分别当 $r \ge \frac{1}{6}, r \ge \frac{1}{4}, r \ge \frac{1}{2}$ 及 $r = \frac{1}{6}$ 时收敛, 且收敛阶为 $O((\Delta t)^2)$.

数值例子表明理论分析是正确的. 因篇幅关系, 从略.

参 考 文 献

1 Douglas J. On the numcrical intergration of $U_{xx} + U_{yy} = U_t$ by implicit methods. J. Soc. Ind. and Appl. Math., 1955, 3: 42～55

2 吴启光. 二维抛物型方程某些差分格式的收敛性和稳定性. 工科数学, 1987, 3(3): 1～8

3 吴启光. 一个新的差分格式的稳定性和收敛性. 计算数学, 1988, 10(4): 388～397

4 曾文平. 解二维抛物型方程的两个高精度显式差分格式. 计算物理, 1992, 9(4): 448～450

A Family of Steady and High Accurate Difference Schemes for Solving Two-Dimensional Equations of Parabolic Type

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Abstract A family of absolutely stable and high accurate difference schemes containing parameter are set up for solving two-dimensional equations of parabolic type. And then, an explicit difference scheme (17) is obtained under the special condition of $\theta = 0$, $r = 1/6$. All these schemes are absolutely stable for the arbitrarily chosen parameters $\theta \le 1/6$, and their convergence order equals to $O((\Delta t)^2)$ in case $0 \le \theta \le \min(1/6, 1/2 - 1/12r)$.

Keywords two-dimensional equations of parabolic type, difference scheme, high accuracy, absolutely stable