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# 解四阶抛物型方程高精度恒稳的隐式格式*

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**摘要** 对四阶抛物型方程 $u_t+u_{xxxx}=0$ 构造了一类三层隐式差分格式．它们含有非负参数 $\alpha_1,\alpha_2$ 和 $\alpha_3$，其局部截断误差至少是 $O(\Delta t^2+\Delta t^6)$．在条件 $\alpha_1\geqslant\alpha_3\geqslant 0, 0\leqslant\alpha_2\leqslant\frac{1}{2}$ 及 $\alpha_1+\alpha_2+\alpha_3=1$ 之下，该格式绝对稳定且可用追赶法求解．

**关键词** 四阶抛物型方程，绝对稳定，高精度，隐式差分格式

**分类号** O 241.82

近来对高阶抛物型方程和方程组的研究逐渐增多．1960 年，Саульев[1]曾对四阶抛物型方程 $u_t+u_{xxxx}=0$ 构造了两个隐式格式，其截断误差分别为 $O(\Delta t^2+\Delta x^2)$ 和 $O(\Delta t^2+\Delta x^4)$．本文构造了一族三层(特殊情况下为两层)含参数、高精度、绝对稳定、五对角线型的隐式差分格式，其截断误差至少为 $O(\Delta t^2+\Delta x^6)$(特殊情况下还可提高)．它比同类隐式格式高二阶或四阶(空间方向)，包含了文〔2〕中的高精度恒稳格式，当参数 $\alpha_1=\alpha_2=\frac{1}{2}$ 和 $\alpha_3=0$ 时可得到一个两层 10 点的高精度恒稳格式．

## 1 差分格式的构造

四阶抛物型方程的初边值问题

$$\left.\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial^4 u}{\partial x^4}=0 \qquad (0<x<1, 0<t\leqslant T),\\ &u(x,0)=f(x) \qquad (0\leqslant x\leqslant 1),\\ &u(0,t)=\frac{\partial^2 u}{\partial x^2}(0,t)=u(1,t)=\frac{\partial^2 u}{\partial x^2}(1,t)=0 \qquad (0\leqslant t\leqslant T).\end{aligned}\right\} \tag{1}$$

当 $f(x)$ 足够光滑时，问题(1)的解存在且唯一，并可表示为级数形式．现在设问题(1)的解 $u(x,t)$ 充分光滑．设时间步长为 $\Delta t$，空间步长为 $\Delta x$，网格区域由点集 $(x_m,t_n)$ $(m=0,1,\cdots,M;n=0,1,\cdots)$ 所组成，其中 $x_m=m\Delta x, t_n=n\Delta t, \Delta x=M^{-1}$．设 $r=\Delta t/\Delta x^4$ 为网格比．在网点 $(x_m,t_n)$ 处的网格函数 $u(x_m,t_n)$ 记为 $u_m^n$．

利用 Taylor 展开不难验证下列数值微分公式

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$$\left(\frac{\partial u}{\partial t}\right)_m^{n+1}=\frac{1}{2\Delta t}(3u_m^{n+1}-4u_m^n+u_m^{n-1})+\frac{\Delta t^2}{3}\left(\frac{\partial^3 u}{\partial t^3}\right)_m^{n+1}+\cdots \tag{2}$$

$$\left(\frac{\partial u}{\partial t}\right)_m^{n}=\frac{1}{2\Delta t}(u_m^{n+1}-u_m^{n-1})-\frac{\Delta t^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)_m^{n}+\cdots \tag{3}$$

$$\left(\frac{\partial u}{\partial t}\right)_m^{n-1}=\frac{1}{2\Delta t}(-u_m^{n+1}+4u_m^n-3u_m^{n-1})+\frac{\Delta t^2}{3}\left(\frac{\partial^3 u}{\partial t^3}\right)_m^{n-1}+\cdots \tag{4}$$

引入记号

$$\Delta_h u_m^n=-u_{m+2}^n+124u_{m+1}^n+474u_m^n+124u_{m-1}^n-u_{m-2}^n, \tag{5}$$

$$L_h u_m^n=u_{m+2}^n-4u_{m+1}^n+6u_m^n-4u_{m-1}^n+u_{m-2}^n, \tag{6}$$

则有

$$\frac{1}{720}\Delta_h u_m^n=u_m^n+\frac{\Delta x^2}{6}\left(\frac{\partial^2 u}{\partial x^2}\right)_m^n+\frac{\Delta x^4}{80}\left(\frac{\partial^4 u}{\partial x^4}\right)_m^n+\frac{7\Delta x^6}{30\ 240}\left(\frac{\partial^6 u}{\partial x^6}\right)_m^n-\frac{11\Delta x^8}{1\ 209\ 600}\left(\frac{\partial^8 u}{\partial x^8}\right)_m^n+O(\Delta x^{10}), \tag{7}$$

$$\frac{1}{\Delta x^4}L_h u_m^n=\left(\frac{\partial^4 u}{\partial x^4}\right)_m^n+\frac{\Delta x^2}{6}\left(\frac{\partial^6 u}{\partial x^6}\right)_m^n+\frac{\Delta x^4}{80}\left(\frac{\partial^8 u}{\partial x^8}\right)_m^n+\frac{17\Delta x^6}{30\ 240}\left(\frac{\partial^{10} u}{\partial x^{10}}\right)_m^n+\frac{31\Delta x^8}{1\ 814\ 400}\left(\frac{\partial^{12} u}{\partial x^{12}}\right)_m^n+O(\Delta x^{10}). \tag{8}$$

于是

$$\frac{1}{720}\Delta_h\left(\frac{\partial^4 u_m^n}{\partial x^4}\right)=\frac{1}{\Delta x^4}L_h u_m^n-\frac{1}{3\ 024}\Delta x^6\left(\frac{\partial^{10} u}{\partial x^{10}}\right)_m^n-\frac{19}{725\ 760}\Delta x^8\left(\frac{\partial^{12} u}{\partial x^{12}}\right)_m^n+O(\Delta x^{10}). \tag{9}$$

将式(9)用于方程

$$\left(\frac{\partial u}{\partial t}\right)_m^k+\left(\frac{\partial^4 u}{\partial x^4}\right)_m^n=0 \qquad (k=n+1,n,n-1), \tag{10}$$

便得下列三个差分格式

$$\frac{1}{720}\Delta_h\frac{3u_m^{n+1}-4u_m^n+u_m^{n-1}}{2\Delta t}+\frac{1}{\Delta x^4}L_h u_m^{n+1}=0, \tag{11}$$

$$\frac{1}{720}\Delta_h\frac{u_m^{n+1}-u_m^{n-1}}{2\Delta t}+\frac{1}{\Delta x^4}L_h u_m^{n}=0, \tag{12}$$

$$\frac{1}{720}\Delta_h\frac{-u_m^{n+1}+4u_m^n-3u_m^{n-1}}{2\Delta t}+\frac{1}{\Delta x^4}L_h u_m^{n-1}=0. \tag{13}$$

它们的局部截断误差分别为

$$R_1=\frac{\tau^2}{2\ 160}\Delta_h\left(\frac{\partial^3 u}{\partial t^3}\right)_m^{n+1}-\frac{\Delta x^6}{3\ 024}\left(\frac{\partial^{10} u}{\partial x^{10}}\right)_m^{n+1}-\frac{19\Delta x^8}{725\ 760}\left(\frac{\partial^{12} u}{\partial x^{12}}\right)_m^{n+1}+O(\Delta x^{10}), \tag{14}$$

$$R_2=\frac{-\tau^2}{4\ 320}\Delta_h\left(\frac{\partial^3 u}{\partial t^3}\right)_m^{n}-\frac{\Delta x^6}{3\ 024}\left(\frac{\partial^{10} u}{\partial x^{10}}\right)_m^{n}-\frac{19\Delta x^8}{725\ 760}\left(\frac{\partial^{12} u}{\partial x^{12}}\right)_m^{n}+O(\Delta x^{10}), \tag{15}$$

$$R_3=\frac{\tau^2}{2\ 160}\Delta_h\left(\frac{\partial^3 u}{\partial t^3}\right)_m^{n-1}-\frac{\Delta x^6}{3\ 024}\left(\frac{\partial^{10} u}{\partial x^{10}}\right)_m^{n-1}-\frac{19\Delta x^8}{725\ 760}\left(\frac{\partial^{12} u}{\partial x^{12}}\right)_m^{n-1}+O(\Delta x^{10}). \tag{16}$$

$\alpha_1\times(11)+\alpha_2\times(12)+\alpha_3\times(13)$便得如下含三参数的三层隐式差分格式为

$$\frac{1}{720}\Delta_h\frac{(3\alpha_1+\alpha_2-\alpha_3)u_m^{n+1}-4(\alpha_1-\alpha_3)u_m^n+(\alpha_1-\alpha_2-3\alpha_3)u_m^{n-1}}{2\Delta t}$$

$$+\frac{1}{\Delta x^4}L_h(\alpha_1 u_m^{n+1}+\alpha_2 u_m^n+\alpha_3 u_m^{n-1})=0, \tag{17}$$

其初、边界条件的离散化处理同文〔1〕. 从略．

当 $\alpha_1+\alpha_2+\alpha_3=1$，注意到公式(7)，并利用等式 $\partial^3 u/\partial t^3=-\partial^{12}u/\partial x^{12}$，便得其相应的局部截断误差为

$$\begin{aligned}R&=\alpha_1 R_1+\alpha_2 R_2+\alpha_3 R_3\\&=-\left(\frac{-2+3\alpha_2}{4320}+\frac{19}{725760r^2}\right)\Delta t^2\left(\frac{\partial^3 u}{\partial t^3}\right)_m^n-\frac{\Delta x^6}{3024}\left(\frac{\partial^{10}u}{\partial x^{10}}\right)_m^n\\&\quad+O(\Delta t^3+\Delta t^2\Delta x^2+\Delta x^{10}).\end{aligned} \tag{18}$$

所以，三层格式(17)当 $\alpha_1+\alpha_2+\alpha_3=1$ 时，其局部截断误差至少为 $O(\Delta t^2+\Delta x^6)$. 若令 $\Delta t^2$ 项的系数为 0，得 $\alpha_2=\frac{2}{3}-\frac{19}{504r^2}$时的局部截断误差为$O(\Delta t^3+\Delta x^6)$. 这时由后面所得稳定性条件之一 $0\leqslant\alpha_2\leqslant\frac{1}{2}$，推得稳定性限制为$\frac{1}{4}\sqrt{\frac{19}{21}}\leqslant r\leqslant\frac{1}{2}\sqrt{\frac{19}{21}}$(当然，还要附加其他稳定性限制 $\alpha_1\geqslant\alpha_3\geqslant0$，$\alpha_2=\frac{2}{3}-\frac{19}{504r^2}$以及 $\alpha_1+\alpha_2+\alpha_3=1$). 但由于 $\Delta x^6$ 项的存在，已不能选取非负参数使差分格式(17)的精度更高而又绝对稳定．

## 2　差分格式稳定性与收敛性

为研究差分格式稳定性，需如下的 Miller 准则．

**引理**[3]　设 $A>0$，实系数二次方程

$$AX^2+BX+C=0$$

的两根按模小于等于 1 的充要条件为

$$A-C\geqslant 0, A+B+C\geqslant 0, A-B+C\geqslant 0.$$

用 Fourier 方法[4]分析差分格式(17)的稳定性．令 $u_m^n=\lambda^n e^{im\theta}$，$|\theta|<\pi$，于是有

$$F\xlongequal{令}\frac{e^{-im\theta}}{720}\Delta_h e^{im\theta}=\frac{1}{180}(119+62\cos\theta-\cos^2\theta)\geqslant\frac{14}{45}>0,$$

$$G\xlongequal{令}re^{-im\theta}L_h e^{im\theta}=4r(\cos\theta-1)^2\geqslant 0.$$

按文〔4〕中理论，格式(17)的传播矩阵为

$$G(m,\Delta t)=\begin{bmatrix}-B/A & -C/A\\ 1 & 0\end{bmatrix}.$$

其特征方程为 $A\lambda^2+B\lambda+C=0$，其中 $A=(3\alpha_1+\alpha_2-\alpha_3)F+2\alpha_1 G$，$B=-4(\alpha_1-\alpha_3)F+2\alpha_2 G$，$C=(\alpha_1-\alpha_2-3\alpha_3)F+2\alpha_3 G$. 注意到 $\alpha_1+\alpha_2+\alpha_3=1$，因此当 $\alpha_1\geqslant\alpha_3\geqslant0$ 时，有 $3\alpha_1+\alpha_2-\alpha_3\geqslant$，$\alpha_1+\alpha_2+\alpha_3=1$ 故 $A\geqslant F+2\alpha_1 G\geqslant F>0$，且 $A-C=2F+2(\alpha_1-\alpha_3)G\geqslant 2F>0$，又 $A+B+C=2G\geqslant0$. 最后，当 $\alpha_1\geqslant\alpha_3\geqslant0$ 且 $\alpha_1+\alpha_3\geqslant\alpha_2\geqslant0$ 时，有 $A-B+C=8(\alpha_1-\alpha_3)F+(\alpha_1-\alpha_2+\alpha_3)G\geqslant0$. 又注意到 $\alpha_1+\alpha_2+\alpha_3=1$，故条件 $\alpha_1+\alpha_3\geqslant\alpha_2\geqslant0$ 等价于 $0\leqslant\alpha_2\leqslant\frac{1}{2}$.

综上所述，当 $\alpha_1\geqslant\alpha_3\geqslant0$，$0\leqslant\alpha_2\leqslant\frac{1}{2}$且 $\alpha_1+\alpha_2+\alpha_3=1$ 时，引理条件成立．由引理结论知此时特征方程的两根按模小于等于 1，且由于 $A-C>0$ 不可能有等于 1 的重根，故差分格式(17)

对任何 $r=\frac{\Delta t}{\Delta x^4}>0$ 均稳定．再由 Lax 稳定性与收敛性等价定理，使得本文如下结论．

**定理**　设 $u(x,t)$ 是四阶抛物型方程(1)的充分光滑的解析解，则当 $\alpha_2\neq\frac{2}{3}-\frac{19}{504r^2}$ 时差分格式(17)逼近方程(1)的局部截断误差为 $O(\Delta t^2+\Delta x^6)$．又当参数 $\alpha_1+\alpha_2+\alpha_3=1,\alpha_1\geqslant\alpha_3\geqslant0$ 且 $0\leqslant\alpha_2\leqslant\frac{1}{2}$ 时，差分格式(17)绝对稳定且收敛．特别地，当 $\alpha_2=\frac{2}{3}-\frac{19}{504r^2}$ 时差分格式(17)逼近方程(1)的局部截断误差高达 $O(\Delta t^3+\Delta x^6)$．这时当其它参数满足 $\alpha_1\geqslant\alpha_3\geqslant0$ 且 $\alpha_1+\alpha_2+\alpha_3=1$ 时，差分格式(17)当 $\frac{1}{4}\sqrt{\frac{19}{21}}\leqslant r\leqslant\frac{1}{2}\sqrt{\frac{19}{21}}$ 时稳定且收敛．

## 3　若干特例

**特例 1**　记 $\alpha_1-\alpha_3=\alpha,\alpha_2=\frac{1}{2}-2\beta$，则由 $\alpha_1+\alpha_2+\alpha_3=1$，推出：(Ⅰ) $3\alpha_1+\alpha_2-\alpha_3=2\alpha+1$；(Ⅱ) $\alpha_1-3\alpha_3-\alpha_2=2\alpha-1$；(Ⅲ) $\alpha_1=\frac{1}{4}+\frac{\alpha}{2}+\beta$；(Ⅳ) $\alpha_3=\frac{1}{4}-\frac{\alpha}{2}+\beta$. 此时格式(17)成为三层 15 点隐式格式：

$$\Delta_h[(2\alpha+1)u_m^{n+1}-4\alpha u_m^n+(2\alpha-1)u_m^{n-1}]$$
$$+1440r[(\frac{1}{4}+\frac{\alpha}{2}+\beta)u_m^{n+1}+(\frac{1}{2}-2\beta)u_m^n+(\frac{1}{4}-\frac{\alpha}{2}+\beta)u_m^{n-1}]=0. \tag{19}$$

这就是文[2]的差分格式

$$-\frac{1}{720\Delta t}[(\alpha+\frac{1}{2})u_{m+2}^{n+1}-2\alpha u_{m+2}^n+(\alpha-\frac{1}{2})u_{m+2}^{n-1}]+\frac{124}{720\Delta t}[(\alpha+\frac{1}{2})u_{m+1}^{n+1}$$
$$-2\alpha u_{m+1}^n+(\alpha-\frac{1}{2})u_{m+1}^{n-1}]+\frac{474}{720\Delta t}[(\alpha+\frac{1}{2})u_m^{n+1}-2\alpha u_m^n+(\alpha-\frac{1}{2})u_m^{n-1}]$$
$$+\frac{124}{720\Delta t}[(\alpha+\frac{1}{2})u_{m-1}^{n+1}-2\alpha u_{m-1}^n+(\alpha-\frac{1}{2})u_{m-1}^{n-1}]-\frac{1}{720\Delta t}[(\alpha+\frac{1}{2})u_{m-2}^{n+1}-2\alpha u_{m-2}^n$$
$$+(\alpha-\frac{1}{2})u_{m-2}^{n-1}]+(\frac{1}{4}+\frac{\alpha}{2}+\beta)\frac{1}{\Delta x^4}[u_{m+2}^{n+1}-4u_{m+1}^{n+1}+6u_m^{n+1}-4u_{m-1}^{n+1}+u_{m-2}^{n+1}]$$
$$+(\frac{1}{2}-2\beta)\frac{1}{\Delta x^4}[u_{m+2}^n-4u_{m+1}^n+6u_m^n-4u_{m-1}^n+u_{m-2}^n]$$
$$+(\frac{1}{4}-\frac{\alpha}{2}+\beta)\frac{1}{\Delta x^4}[u_{m+2}^{n-1}-4u_{m+1}^{n-1}+6u_m^{n-1}-4u_{m-1}^{n-1}+u_{m-2}^{n-1}]=0, \tag{19$'$}$$

它对任何非负实数 $\alpha\geqslant0$ 和 $\beta\geqslant0$ 均稳定．

**特例 2**　当 $\alpha_1=\alpha_2=\frac{1}{2}$ 和 $\alpha_3=0$ 时为两层 10 点恒稳格式

$$\Delta_h(u_m^{n+1}-u_m^n)+360rL_h(u_m^{n+1}+u_m^n)=0, \tag{20}$$

也即

$$-[u_{m+2}^{n+1}+u_{m-2}^{n+1}-124(u_{m+1}^{n+1}+u_{m-1}^{n+1})-474u_m^{n+1}]$$
$$+[u_{m+2}^n+u_{m-2}^n-124(u_{m+1}^n+u_{m-1}^n)-474u_m^n]$$
$$+360r[u_{m+2}^{n+1}-4u_{m+1}^{n+1}+6u_m^{n+1}-4u_{m-1}^{n+1}+u_{m-2}^{n+1}$$
$$+u_{m+2}^n-4u_{m+1}^n+6u_m^n-4u_{m-1}^n+u_{m-2}^n]=0. \tag{20$'$}$$

值得注意的是，当 $\alpha_1=0$ 和 $\alpha_2=\alpha_3=\frac{1}{2}$ 时，格式(17)虽然也是两层格式，但参数取值不满足稳定条件．

**特例 3**　当 $\alpha_1=\alpha_3=\frac{1}{4}$，$\alpha_2=\frac{1}{2}$ 及 $\alpha_1=1$，$\alpha_2=\alpha_3=0$ 时，均是恒稳的三层 15 点格式．

**特例 4**　当 $\alpha_1=\alpha_3=\frac{1}{2}$ 和 $\alpha_2=0$ 时，格式(17)为恒稳的三层 10 点格式．

**特例 5**　当 $\alpha_1=\alpha_3$ 和 $2\alpha_1\geqslant\alpha_2$ 时，记 $\alpha_1=\frac{b_1}{2}$，$\alpha_2=b_0$，显然有 $b_0+b_1=1$，$b_0\leqslant\frac{1}{2}$，格式(17)成为恒稳的三层 11 点格式

$$\Delta_h(u_m^{n+1}-u_m^{n-1})=720r(b_1u_m^{n+1}+2b_0u_m^n+b_1u_m^{n-1}). \tag{21}$$

因篇幅关系，恕不一一详列．

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# A Class of High Accurate and Absolutely Stable Implicit Difference Schemes for Solving Four Order Parabolic Equations


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**Abstract**　A class of three layered implicit difference schemes are constructed for solving four order parabolic equations $U_t+u_{xxxx}=0$. They contain nonnegative parameters $\alpha_1$, $\alpha_2$ and $\alpha_3$, with local truncation errors of $O(\Delta t^2+\Delta x^6)$ at least. Under the conditions of $\alpha_1+\alpha_2+\alpha_3=1$, $\alpha_1\geqslant\alpha_3\geqslant0$ and $0\leqslant\alpha_2\leqslant1/2$., these schemes are absolutely stable and can be easily solved by double sweeping method.


**Keywords**　four order parabolic equation, absolutely stable, high accurate, implicit difference scheme