

解多维抛物型方程的两个三层显格式*

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摘要 提出两类解 N -维抛物型方程的两个三层显式差分格式. 显式格式(I)的稳定性条件是 $\max_{1 \leq p \leq N} \gamma_p \leq \frac{1}{2N}$, 且 $\sum_{p=1}^N \gamma_p \neq \frac{1}{2}$, 其中 $\gamma_p = \frac{\Delta t}{(\Delta x_p)^2}$, $p=1, 2, \dots, N$. 而显式格式(II)当参数 $\alpha \geq 1$ 时是无条件稳定的. 它们的局部截断误差阶分别是 $O((\Delta t)^2 + \sum_{p=1}^N (\Delta x_p)^2)$ 及 $O((\Delta t)^2 + \sum_{p=2}^N (\Delta x_p)^2 + \sum_{p=1}^N (\frac{\Delta t}{\Delta x_p})^2)$.

关键词 多维抛物型偏微分方程, 三层显式差分格式, 分离变量法

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在工程物理问题中,经常需要求解下列多维抛物型偏微分方程

$$\frac{\partial u}{\partial t} = a \sum_{p=1}^N \frac{\partial^2 u}{\partial x_p^2} \quad (0 \leq x_p \leq 1, p=1, 2, \dots, N; t > 0), \quad (1)$$

第一边值问题(为简单起见,初始及边界条件从略,下同). 其中 x_1, x_2, \dots, x_N 为空间变量, t 为时间变量, $a > 0$ 为已知常数. u 为 x_1, x_2, \dots, x_N, t 的未知函数.

文[1]针对此问题提出一个两层显式格式,但局部截断误差阶仅为 $O((\Delta t) + (\Delta x)^2)$, 稳定性条件为 $\gamma = \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2N}$, 也十分苛刻. 为了解决文[1]中显式格式所存在的缺点,我们提出了两类求解其数值解的简单而实用的三层显式差分格式. 格式(I)是条件稳定的,其稳定性条件为 $\max_{1 \leq p \leq N} \gamma_p \leq \frac{1}{2N}$, 且 $\sum_{p=1}^N \gamma_p \neq \frac{1}{2}$, 然而关于在 Δt 上的局部截断误差却高于文[1]中显式格式,其阶为 $O((\Delta t)^2 + \sum_{p=1}^N (\Delta x_p)^2)$, 它是文[2]中的二维格式对高维方程的推广;而格式(II)当参数 $\alpha \geq 1$ 时是无条件稳定的,其局部截断误差为 $O((\Delta t)^2 + \sum_{p=1}^N (\Delta x_p)^2 + \sum_{p=1}^N (\frac{\Delta t}{\Delta x_p})^2)$, 它是 Du Fort Frankel 格式的推广.

令 Δt 为时间步长, Δx_p 是空间 x_p 方向的步长 ($p=1, 2, \dots, N$), 且设 $\Delta x_p = \frac{1}{M_p}$, 网格比 $r_p = \frac{\Delta t}{(\Delta x_p)^2}$ ($p=1, 2, \dots, N$). 同时引入下列记号:

$$u^k = u_{j_1, \dots, j_N}^k = u(j_1 \Delta x_1, \dots, j_N \Delta x_N, k \Delta t),$$

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$$\Delta u^k = (\Delta u^k)_{j_1, \dots, j_N} = u_{j_1, \dots, j_N}^{k+1} - u_{j_1, \dots, j_N}^k,$$

$$\delta_{x_p}^2 u^k = (\delta_{x_p}^2 u^k)_{j_1, \dots, j_N} = u_{j_1, \dots, j_{p-1}, j_p+1, j_{p+1}, \dots, j_N}^k - 2u_{j_1, \dots, j_p, \dots, j_N}^k + u_{j_1, \dots, j_{p-1}, j_p-1, j_{p+1}, \dots, j_N}^k.$$

假定 u 具有所需要的阶的有界连续偏导数, 并为简单起见, 在推导局部截断误差时, 把 $u^k = u_{j_1, \dots, j_N}^k = u_{j_1, \dots, j_p, \dots, j_N}^k$ 理解为节点 $(j_1 \Delta x_1, \dots, j_p \Delta x_p, \dots, j_N \Delta x_N, k \Delta t)$ 处的 u 值.

为证明差分格式的稳定性, 我们需要如下引理.

引理^[3] 实系数二次方程

$$\lambda^2 - b\lambda - c = 0$$

的根按模 ≤ 1 的充要条件为

$$|b| \leq 1 - c \leq 2.$$

1 三层显式格式(I)

首先建立一个三层含参数的差分格式

$$\begin{aligned} \eta_1 \frac{(\Delta u^k)_{j_1, \dots, j_p, \dots, j_N}}{\Delta t} + \eta_2 \frac{\sum_{p=1}^N \{(\Delta u^{k-1})_{j_1, \dots, j_p+1, \dots, j_N} + \Delta u^{k-1}_{j_1, \dots, j_p-1, \dots, j_N}\}}{\Delta t} \\ = \eta_3 a \sum_{p=1}^N \frac{1}{(\Delta x_p)^2} \delta_{x_p}^2 u_{j_1, \dots, j_p, \dots, j_N}^k, \end{aligned} \quad (2)$$

其中 η_1, η_2, η_3 为待定参数.

利用下列数值微分公式

$$\begin{aligned} \frac{1}{\Delta t} \Delta u_{j_1, \dots, j_p, \dots, j_N}^k &= \left(\frac{\partial u}{\partial t}\right)_{j_1, \dots, j_p, \dots, j_N}^k + \frac{\Delta t}{2} \left(\frac{\partial^2 u}{\partial t^2}\right)_{j_1, \dots, j_p, \dots, j_N}^k + O((\Delta t)^2), \\ \frac{1}{\Delta t} \Delta u_{j_1, \dots, j_p \pm 1, \dots, j_N}^{k-1} &= \left(\frac{\partial u}{\partial x}\right)_{j_1, \dots, j_p, \dots, j_N}^k - \frac{1}{2} \Delta t \left(\frac{\partial^2 u}{\partial x^2}\right)_{j_1, \dots, j_p, \dots, j_N}^k \\ &\quad \pm (\Delta x_p) \left(\frac{\partial^3 u}{\partial x_p \partial t}\right)_{j_1, \dots, j_p, \dots, j_N}^k + O((\Delta t)^2 + \Delta t (\Delta x_p)^2), \end{aligned}$$

代入式(2)经整理得

$$\begin{aligned} (\eta_1 + 2N\eta_2) \left(\frac{\partial u}{\partial t}\right)_{j_1, \dots, j_p, \dots, j_N}^k + \left(\frac{1}{2}\eta_1 - N\eta_2\right) \Delta t \left(\frac{\partial^2 u}{\partial t^2}\right)_{j_1, \dots, j_p, \dots, j_N}^k \\ = \eta_3 a \sum_{p=1}^N \left(\frac{\partial^2 u}{\partial x_p^2}\right)_{j_1, \dots, j_p, \dots, j_N}^k + O((\Delta t)^2 + \sum_{p=1}^N (\Delta x_p)^2). \end{aligned} \quad (3)$$

由此可见, 当下列条件成立时

$$\left. \begin{aligned} \eta_1 + 2N\eta_2 &= 1, \\ \frac{1}{2}\eta_1 - N\eta_2 &= 0, \\ \eta_3 &= 1, \end{aligned} \right\} \quad (4)$$

差分格式(2)逼近于方程(1)的局部截断误差阶为 $O((\Delta t)^2) + \sum_{p=1}^N (\Delta x_p)^2$.

解方程组(4), 可以得到 $\eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{4N}, \eta_3 = 1$. 于是, 我们得局部截断误差阶为 $(\Delta t)^2 +$

$\sum_{p=1}^N (\Delta x_p)^2$ 的如下三层显式差分格式(I):

$$\frac{(\Delta u^k)_{j_1, \dots, j_p, \dots, j_N}}{2\Delta t} + \frac{1}{4N} \frac{\sum_{p=1}^N \Delta (u_{j_1, \dots, j_p+1, \dots, j_N}^{k-1} + u_{j_1, \dots, j_p-1, \dots, j_N}^{k-1})}{\Delta t} = a \sum_{p=1}^N \frac{\delta_{x_p}^2 u_{j_1, \dots, j_p, \dots, j_N}^k}{(\Delta x_p)^2}. \quad (5)$$

或令 $r_p = a\Delta t / (\Delta x_p)^2$ ($p=1, 2, \dots, N$), 则式(5)可改写为

$$u_{j_1, \dots, j_p, \dots, j_N}^{k+1} = (1 - 4 \sum_{p=1}^N r_p) u_{j_1, \dots, j_p, \dots, j_N}^k + \frac{1}{2N} \sum_{p=1}^N (u_{j_1, \dots, j_p+1, \dots, j_N}^k + u_{j_1, \dots, j_p-1, \dots, j_N}^k) + \frac{1}{2N} \sum_{p=1}^N (u_{j_1, \dots, j_p+1, \dots, j_N}^{k-1} + u_{j_1, \dots, j_p-1, \dots, j_N}^{k-1}). \quad (6)$$

以下统称差分格式(5)或(6)为三层显式格式(I).

用分离变量法分析格式(I)的稳定性. 令

$$u_{j_1, \dots, j_N}^k = \rho^k e^{i(\sum_{p=1}^N j_p \theta_p)} \quad (i = \sqrt{-1}, |\theta_p| < \pi), \quad (7)$$

代入式(5)并整理得特征方程为

$$\rho^2 - (1 - \frac{1}{N} \sum_{p=1}^N \cos \theta_p - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2}) \rho - \frac{1}{N} \sum_{p=1}^N \cos \theta_p = 0, \quad (8)$$

或

$$\rho^2 - (\frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2} - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2}) \rho - (1 - \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2}) = 0, \quad (9)$$

对照引理, 显然有 $c = 1 - \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2}$, $1 - c = \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2} \leq 2$,

$$|b| = |\frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2} - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2}| \leq 1 - c = \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2},$$

即

$$- \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2} \leq \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2} - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2} \leq \frac{2}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2}. \quad (10)$$

式(10)右端不等式显然对任意 $r_p > 0$ ($p=1, 2, \dots, N$) 均成立, 而左端不等式即为

$$8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2} \leq \frac{4}{N} \sum_{p=1}^N \sin^2 \frac{\theta_p}{2}, \text{ 或 } \sum_{p=1}^N (2r_p - \frac{1}{N}) \sin^2 \frac{\theta_p}{2} \leq 0,$$

它当 $r_p \leq \frac{1}{2N}$ ($p=1, 2, \dots, N$) 或 $\max_{1 \leq p \leq N} r_p \leq \frac{1}{2N}$ 时恒成立, 这也是式(10)成立的条件. 由引理知,

当 $\max_{1 \leq p \leq N} r_p \leq \frac{1}{2N}$ 时, 特征方程(9)的两根按模 ≤ 1 .

又由直接验证可知, 当 $\sum_{p=1}^N r_p \neq \frac{1}{2}$ 时, 可保证 $\rho = \pm 1$ 不是特征方程(9)的重根.

综上所述, 根据分离变量法的一般理论, 我们有

定理 2 当 $\max_{1 \leq p \leq N} r_p \leq \frac{1}{2N}$ 且 $\sum_{p=1}^N r_p \neq \frac{1}{2}$ 时, 三层显格式(I) (即式(5)或式(6)) 稳定.

注 1 当网格取等步长 $\Delta x_1 = \Delta x_2 = \dots = \Delta x_N = \Delta x$ 时, 网格比 $r_1 = r_2 = \dots = r_N = r$, 此时差分格式(I)的稳定性条件为 $r < \frac{1}{2N}$

注 2 文[2]中的三层显式格式为本文格式(I)当 $N=2$ 时的特例.

2 三层显式格式(Ⅱ)

为克服格式(Ⅰ)稳定性条件限制过于苛刻的缺点,把耗散项 $\epsilon \frac{\mathcal{F}u}{\alpha^2}$ 加进高维方程(1)得

$$\frac{\partial u}{\partial t} = a \sum_{p=1}^N \frac{\mathcal{F}u}{\alpha x_p^2} + \epsilon \frac{\mathcal{F}u}{\alpha^2}, \quad (11)$$

令 $\epsilon = -\tau \alpha \sum_{p=1}^N r_p$, 并利用下列数值微分公式:

$$\frac{1}{2\tau}(u_{j_1, \dots, j_N}^{t+1} - u_{j_1, \dots, j_N}^{t-1}) = \left(\frac{\partial u}{\partial t}\right)_{j_1, \dots, j_N}^t + O((\Delta t)^2),$$

$$\frac{1}{(\Delta x_p)^2} \delta_{x_p}^2 u_{j_1, j_p, \dots, j_N}^t = \left(\frac{\mathcal{F}u}{\alpha x_p^2}\right)_{j_1, \dots, j_N}^t + O((\Delta x_p)^2),$$

$$\frac{1}{(\Delta t)^2}(u_{j_1, \dots, j_N}^{t+1} - 2u_{j_1, \dots, j_N}^t + u_{j_1, \dots, j_N}^{t-1}) = \left(\frac{\partial^2 u}{\partial t^2}\right)_{j_1, \dots, j_N}^t + O((\Delta t)^2).$$

将上述各式代入方程(1),并舍去局部截断误差 $O((\Delta t)^2 + \sum_{p=1}^N (\Delta x_p)^2 + \sum_{p=1}^N (\frac{\Delta t}{\Delta x_N})^2)$. 便得三层显式格式(Ⅱ)如式(12)或式(13)

$$\begin{aligned} & \frac{1}{2\tau}(u_{j_1, \dots, j_N}^{t+1} - u_{j_1, \dots, j_N}^{t-1}) \\ &= \sum_{p=1}^N r_p \delta_{x_p}^2 u_{j_1, \dots, j_p, \dots, j_N}^t - \alpha \frac{1}{2\tau} \left(\sum_{p=1}^N r_p\right) (u_{j_1, \dots, j_N}^{t+1} - 2u_{j_1, \dots, j_N}^t + u_{j_1, \dots, j_N}^{t-1}), \end{aligned} \quad (12)$$

或者

$$\begin{aligned} & (1 + 2\alpha \sum_{p=1}^N r_p) u_{j_1, \dots, j_N}^{t+1} = 4(\alpha - 1) \left(\sum_{p=1}^N r_p\right) u_{j_1, \dots, j_N}^t \\ & + 2 \sum_{p=1}^N r_p (u_{j_1, \dots, j_{p+1}, \dots, j_N}^t + u_{j_1, \dots, j_{p-1}, \dots, j_N}^t) + (1 - 2\alpha \sum_{p=1}^N r_p) u_{j_1, \dots, j_N}^{t-1}. \end{aligned} \quad (13)$$

当 $\Delta t = O((\Delta x_p)^2)$, ($p=1, 2, \dots, N$) 时, 差分格式(Ⅱ)(即式(12)或(13))相容逼近于方程(1).

由分离变量法知, 差分格式(Ⅱ)的特征方程为

$$(1 + 2\alpha \sum_{p=1}^N r_p) \rho^2 - (4\alpha \sum_{p=1}^N r_p - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2}) \rho - (1 - 2\alpha \sum_{p=1}^N r_p) = 0. \quad (14)$$

对照引理得, 当 $\alpha > 0, r_p > 0$ ($p=1, 2, \dots, N$) 时, 有

$$-1 < c = \frac{1 - 2\alpha \sum_{p=1}^N r_p}{1 + 2\alpha \sum_{p=1}^N r_p} < 1,$$

且

$$1 - c = \frac{4\alpha \sum_{p=1}^N r_p}{1 + 2\alpha \sum_{p=1}^N r_p} \leq 2,$$

又

$$|b| = \left| \frac{4\alpha \sum_{p=1}^N r_p - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2}}{1 + 2\alpha \sum_{p=1}^N r_p} \right| \leq 1 - c = \frac{4\alpha \sum_{p=1}^N r_p}{1 + 2\alpha \sum_{p=1}^N r_p}.$$

当 $\alpha > 0$ 时, 上式即为

$$-4\alpha \sum_{p=1}^N r_p \leq 4\alpha \sum_{p=1}^N r_p - 8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2} \leq 4\alpha \sum_{p=1}^N r_p. \quad (15)$$

式(15)右端自然成立,而左端即为

$$8 \sum_{p=1}^N r_p \sin^2 \frac{\theta_p}{2} \leq 8\alpha \sum_{p=1}^N r_p,$$

它当 $\alpha \geq 1$ 时恒成立,即当 $\alpha \geq 1$ 时式(15)恒成立. 由引理知特征方程(14)的两根按模 ≤ 1 . 又因 $|c| < 1$, 由韦达定理知至少有一根 < 1 .

于是根据分离变量法的一般理论我们有

定理 2 当 $\alpha \geq 1$ 时, 三层显式格式 (I) (即式(12)或式(13)) 是绝对稳定的.

特别地, 当 $\alpha = 1$ 时差分格式 (I) 即为 Du Fort Frankel 格式. 因此, 定理 1 包含了 Du Fort Frankel 格式.

最后应该指出, 本文所构造的两个格式都是三层显式格式, 除初始值已知外, 尚需补充计算第一层的网格函数值 u_{j_1, \dots, j_N}^1 . 为保证计算精度, 可考虑使用 Crank-Nicolson 格式来计算 u_{j_1, \dots, j_N}^1 .

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Two Classes of Explicit Schemes for Solving Multi-Dimensional Parabolic Equations

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Abstract Two classes of explicit three-level difference schemes are derived for solving parabolic partial differential equation of N-dimension. Explicit scheme I is stable under the condition of $\frac{\max_{1 \leq p \leq N} r_p}{2} \leq \frac{1}{2N}$ and $\sum_{p=1}^N r_p \neq \frac{1}{2}$, where $r_p = \Delta t / (\Delta r_p)^2$, ($p = 1, 2, \dots, N$); while explicit scheme II is unconditionally stable in case parameter $\alpha \geq 1$. Their respective local truncation error are $O((\Delta t)^2 + \sum_{p=1}^N (\Delta r_p)^2)$ and $O((\Delta t)^2 + \sum_{p=1}^N (\Delta r_p)^2 + \sum_{p=1}^N (\frac{\Delta t}{\Delta r_p})^2)$.

Keywords multi dimension parabolic partial differential equation, three-level difference scheme, separation of variables