解三维抛物型方程的高精度显式格式*

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摘要　提出解三维抛物型方程的两层以及三层的高精度显式差分格式.它们的局部截断误差都是 $O(\Delta t^2)$，而稳定性条件分别为 $r=1/6$ 和 $r<1/6$.

关键词　三维抛物型方程，高精度，显式差分格式

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在渗流、扩散、热传导等许多领域经常会遇到求解抛物型方程的问题.在三维情形，其模型为如下初边值问题

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\quad(0\leqslant x,y,z\leqslant 1,t>0),\\
&u(0,y,z,t)=g_1(y,z,t),\ u(1,y,z,t)=g_2(y,z,t)\quad(0\leqslant y,z\leqslant 1,t>0),\\
&u(x,0,z,t)=h_1(x,z,t),\ u(x,1,z,t)=h_2(x,z,t)\quad(0\leqslant x,z\leqslant 1,t>0),\\
&u(x,y,0,t)=\xi_1(x,y,t),\ u(x,y,1,t)=\xi_2(x,y,t)\quad(0\leqslant x,y\leqslant 1,t>0),\\
&u(x,y,z,0)=f(x,y,z)\quad(0\leqslant x,y,z\leqslant 1).
\end{aligned}\right\}\tag{1}$$

解上述方程的古典显式或隐式格式或 Crank-Nicholson 格式的精度均不高，截断误差仅为 $O(\Delta t+(\Delta x)^2)$ 或 $O((\Delta t)^2+(\Delta x)^2)$.在文〔2〕中构造了精度较高且绝对稳定的差分格式，其截断误差达到 $O((\Delta t)^2+(\Delta x)^4))$，但却是三层隐式格式，徒增了很多计算量与存储量.本文将构造一个三层显格式，其精度与文〔2〕格式相同，即截断误差达到 $O((\Delta t)^2+(\Delta x)^4))$，而稳定性条件为 $r=\Delta t/(\Delta x)^2=\Delta t/(\Delta y)^2=\Delta t/(\Delta z)^2<1/6$.另外，我们也给出一类高精度恒稳的二层隐式格式，特殊情况下得到一个两层的高精度显式格式，其 $r=1/6$ 明显地优于古典显式格式(低精度、稳定性条件为 $r\leqslant 1/6$).

1　高精度差分格式的构造

设 Δt 为时间步长，$\Delta x,\Delta y,\Delta z$ 分别为 x,y,z 方向的空间步长，且为简便计，设 $\Delta x=\Delta y=\Delta z=1/M$ (M 为正整数) 用如下含参数的差分方程逼近微分方程(1).

$$\{\eta_0\Delta\omega^n_{i,j,k}+\eta_1\Delta(\Box+4)\omega^{n-1}_{i,j,k}+\eta_2\Delta(\Box+4)\omega^{n-1}_{i,j,k}+\eta_3\Delta\omega^{n-1}_{i,j,k}+\eta_4\nabla\omega^n_{i,j,k}\}/\Delta t$$
$$=\frac{1}{(\Delta x)^2}\{\frac{\theta_1}{4}\Box\omega^n_{i,j,k}+\frac{\theta_2}{2}\Diamond\omega^n_{i,j,k}\}+\frac{1}{(\Delta x)^2}\{\frac{\theta_3}{4}\Box\omega^{n-1}_{i,j,k}+\frac{\theta_4}{2}\Diamond\omega^{n-1}_{i,j,k}\},\tag{2}$$

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其中 $\omega^n_{i,j,k}$ 表示在节点 $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ 处的网格函数值，且记

$$\left.\begin{aligned}
&\Delta\omega^n_{i,j,k}=\omega^{n+1}_{i,j,k}-\omega^n_{i,j,k},\ \nabla\omega^n_{i,j,k}=\omega^{n+1}_{i,j,k}-\omega^{n-1}_{i,j,k},\\
&\Box\omega^n_{i,j,k}=(z^{\Box}+y^{\Box}+x^{\Box})\omega^n_{i,j,k},\\
&\Diamond\omega^n_{i,j,k}=(z\Diamond+y\Diamond+x\Diamond)\omega^n_{i,j,k},\\
&z^{\Box}\omega^n_{i,j,k}=\omega^n_{i+1,j+1,k}+\omega^n_{i-1,j+1,k}+\omega^n_{i+1,j-1,k}+\omega^n_{i-1,j-1,k}-4\omega^n_{i,j,k},\\
&z^{\Diamond}\omega^n_{i,j,k}=\omega^n_{i+1,j,k}+\omega^n_{i,j+1,k}+\omega^n_{i-1,j,k}+\omega^n_{i,j-1,k}-4\omega^n_{i,j,k}.
\end{aligned}\right\}\tag{3}$$

其余类推. $\eta_k(k=0\sim4)$ 及 $\theta_k(k=1\sim4)$ 为待定参数. 适当选择这些参数，可以使差分格式(2)逼近微分方程(1)时，不仅具有尽可能高阶的离散误差，而且有较好的稳定性.

当微分方程(1)的解充分光滑时，有如下关系式成立

$$\frac{\partial^n}{\partial t^n}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^m\omega=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)^{m+2n}\omega,\tag{4}$$

其中 m,n 为非负整数. 在节点 $(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$ 处进行 Taylor 展开得

$$\frac{1}{2(\Delta x)^2}z^{\Box}\omega^n_{i,j,k}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\omega^n_{i,j,k}+\frac{(\Delta x)^2}{12}\left[\frac{\partial^4}{\partial x^4}+6\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right]\omega^n_{i,j,k}+0((\Delta x)^4)\tag{5}$$

等等，及

$$\begin{aligned}
\frac{1}{4(\Delta x)^2}\Box\omega^n_{i,j,k}&=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\omega^n_{i,j,k}+\frac{(\Delta x)^2}{12}\left[\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}+\frac{\partial^4}{\partial z^4}\right)\right.\\
&\quad\left.+3\left(\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^2\partial z^2}+\frac{\partial^4}{\partial z^2\partial x^2}\right)\right]\omega^n_{i,j,k}+0((\Delta x)^4)\\
&=\frac{\partial\omega^n_{i,j,k}}{\partial t}+\frac{(\Delta x)^2}{12}\frac{\partial^2\omega^n_{i,j,k}}{\partial t^2}+\frac{(\Delta x)^2}{12}\left[\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^2\partial z^2}+\frac{\partial^4}{\partial z^2\partial x^2}\right]\omega^n_{i,j,k}+0((\Delta x)^4),
\end{aligned}\tag{6}$$

$$\frac{1}{(\Delta x)^2}z^{\Diamond}\omega^n_{i,j,k}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\omega^n_{i,j,k}+\frac{(\Delta x)^2}{12}\left(\frac{\partial^4}{\partial x^4}+\frac{\partial^4}{\partial y^4}\right)\omega^n_{i,j,k}+0((\Delta x)^4)\tag{7}$$

等等，及

$$\begin{aligned}
\frac{1}{2(\Delta x)^2}\Diamond\omega^n_{i,j,k}&=\frac{\partial\omega^n_{i,j,k}}{\partial t}+\frac{(\Delta x)^2}{12}\frac{\partial^2\omega^n_{i,j,k}}{\partial t^2}-\frac{2(\Delta x)^2}{12}\left[\frac{\partial^4}{\partial x^2\partial y^2}\right.\\
&\quad\left.+\frac{\partial^4}{\partial y^2\partial z^2}+\frac{\partial^4}{\partial z^2\partial x^2}\right]+0((\Delta x)^4),
\end{aligned}\tag{8}$$

将它们代入差分方程(2)，并使用关系式(5)可得

$$\begin{aligned}
&(\eta_0+4\eta_1+4\eta_2+\eta_3+2\eta_4)\frac{\partial\omega^n_{i,j,k}}{\partial t}+\left(\frac{\eta_0}{2}-2\eta_1-2\eta_2-\frac{\eta_3}{2}\right)\Delta t\frac{\partial^2\omega^n_{i,j,k}}{\partial t^2}\\
&+(2\eta_1+4\eta_2)(\Delta x)^2\frac{\partial^2\omega^n_{i,j,k}}{\partial t^2}-(\eta_1+2\eta_2)(\Delta x)^2\Delta t\frac{\partial^3\omega^n_{i,j,k}}{\partial t^3}+0((\Delta t)^2+(\Delta x)^4)\\
&=(\theta_1+\theta_2+\theta_3+\theta_4)\frac{\partial\omega^n_{i,j,k}}{\partial t}-(\theta_3+\theta_4)\Delta t\frac{\partial^2\omega^n_{i,j,k}}{\partial t^2}+\frac{(\Delta x)^2}{12}(\theta_1+\theta_2+\theta_3+\theta_4)\frac{\partial^2\omega^n_{i,j,k}}{\partial t^2}\\
&+\frac{(\Delta x)^2}{12}[(\theta_1+\theta_3)-2(\ \ +\theta_4)]\left[\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^2\partial z^2}+\frac{\partial^4}{\partial z^2\partial x^2}\right]\omega^n_{i,j,k}\\
&-\frac{\Delta t(\Delta x)^2}{12}(\theta_3+\theta_4)\frac{\partial^3\omega^n_{i,j,k}}{\partial t^3}-\frac{1}{12}(\theta_3-2\theta_4)\Delta t(\Delta x)^2\\
&\times\frac{\partial}{\partial t}\left[\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^2\partial z^2}+\frac{\partial^4}{\partial z^2\partial x^2}\right]\omega^n_{i,j,k}+0((\Delta t)^2+(\Delta x)^4),
\end{aligned}\tag{9}$$

上式中的 $\theta_1+\theta_2+\theta_3+\theta_4$，可以是任何非零常数，为了使误差阶达到 $0((\Delta t)^2+(\Delta x)^4)$，只需下

列诸方程同时成立

$$\left.\begin{aligned}&\theta_1+\theta_2+\theta_3+\theta_4=1,\eta_0+4(\eta_1+\eta_2)+\eta_3+2\eta_4=1,\\&2\eta_1+4\eta_2=1/12,\theta_1+\theta_3=2(\theta_2+\theta_4),\\&(\eta_0-\eta_3)/2-2(\eta_1+\eta_2)=-(\theta_3+\theta_4),\\&\eta_1+2\eta_2=(\theta_3+\theta_4)/12,\theta_3=2\theta_4.\end{aligned}\right\}\tag{10}$$

解方程组(10)得

$$\left.\begin{aligned}&\theta_1=\theta_3=1/3,\quad\theta_2=\theta_4=1/6,\\&\eta_0=\eta-5/6-4\omega,\ \eta_1=1/24-2\omega,\ \eta_2=\omega,\ \eta_3=\eta,\ \eta_4=5/6-\eta+4\omega.\end{aligned}\right\}\tag{11}$$

将式(11)代入式(2),整理得两层差分格式为

$$\begin{aligned}&\{1+(1/24-r/12-2\omega)\Diamond+(\omega-r/12)\Box\}\omega^n_{i,j,k}\\&=\{1+(1/24+r/12-2\omega)\Diamond+(\omega+r/12)\Box\}\omega^{n-1}_{i,j,k},\end{aligned}\tag{12}$$

其中 $r=\Delta t/(\Delta x)^2=\Delta t/(\Delta y)^2=\Delta t/(\Delta z)^2$,这是一个隐式格式,其截断误差为 $0((\Delta t)^2+(\Delta x)^4)$. 我们指出差分格式(12)的几个特例,由后面的稳定性分析可知,这几个特殊格式都是稳定的.

特例1 当 $\omega=0$ 时,格式(12)成为

$$\begin{aligned}(\omega^n_{i,j,k}-\omega^{n-1}_{i,j,k})/\Delta t=&[(1/12-1/24r)\Diamond+\Box 1/12]\omega^n_{i,j,k}\\&+[(1/12+1/24r)\Diamond+\Box 1/12]\omega^n_{i,j,k}.\end{aligned}\tag{13}$$

特例2 当 $\omega=1/72,r=1/6$ 时,格式(12)成为两层的高精度显式格式

$$\omega^n_{i,j,k}=\{1+1/36\Diamond+1/36\Box\}\omega^{n-1}_{i,j,k}.\tag{14}$$

如果将式(10)中最后二个式子去掉,则得

$$\left.\begin{aligned}&\theta_1=\frac{2}{3}-\theta,\ \theta_2=\frac{1}{3}-\varphi,\theta_3=\theta,\ \theta_4=\varphi,\\&\eta_0=\frac{1}{6}+\eta-4\omega-2(\theta-\varphi),\eta_1=\frac{1}{24}-2\omega,\\&\eta_2=\omega,\ \eta_3=\eta,\ \eta_4=\frac{1}{3}-\eta+4\omega+(\theta+\varphi).\end{aligned}\right\}\tag{15}$$

代入式(2)则得如下差分格式

$$\begin{aligned}&[\frac{1}{2}-(\theta+\varphi)]\omega^{n+1}_{i,j,k}=[\frac{r}{4}(2/3-\theta)-\omega]\Box\omega^n_{i,j,k}\\&+[\frac{r}{2}(\frac{1}{3}-\varphi)-(\frac{1}{24}-2\omega)]\Diamond\omega^n_{i,j,k}-2(\theta+\varphi)\omega^n_{i,j,k}\\&+[\frac{r}{4}\theta+\omega]\Box\omega^{n-1}_{i,j,k}+[\frac{r}{2}\varphi+(\frac{1}{24}-2\omega)]\Diamond\omega^{n-1}_{i,j,k}\\&+[\frac{1}{2}-(\theta+\varphi)]\omega^{n-1}_{i,j,k},\end{aligned}\tag{16}$$

这是一个三层显式格式,其截断误差为 $0((\Delta t)^2+\Delta t(\Delta x)^2+(\Delta x)^4)$.

2 稳定性分析

定理1 当 $\omega\geqslant 0$ 时,两层隐式格式(12)绝对稳定. 特别地,当 $\omega=1/72$、$r=1/6$ 时,两层

显式格式(14)稳定.

证明 根据分离变量法,令

$$\omega_{i,j,k}^{n}=\omega^{n}e^{i^{*}(p\pi xi+q\pi yi+l\pi zk)},\quad (i^{*}=\sqrt{-1}),\tag{17}$$

且记

$$\left.\begin{aligned}&\Diamond\omega_{i,j,k}^{n}=-8(s_1^2+s_2^2+s_3^2)\omega_{i,j,k}^{n},\\&\Box\omega_{i,j,k}^{n}=\{-16(s_1^2+s_2^2+s_3^2)+16(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)\}\omega_{i,j,k}^{n},\\&s_1=\sin\frac{p\pi\Delta x}{2},s_2=\sin\frac{q\pi\Delta y}{2},s_3=\sin\frac{l\pi\Delta z}{2}.\end{aligned}\right\}\tag{18}$$

代入式(12)可得传播因子 $G=(A-B)/(A+B)$,其中 $A=1-\frac{1}{3}(s_1^2+s_2^2+s_3^2)+16\omega(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)$,$B=2r\{(s_1^2+s_2^2+s_3^2)-\frac{2}{3}(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)\}$. 易知,对任意 $r>0$ 均有 $B\geqslant0$,且当 $\omega\geqslant0$ 时,$A>0$. 因而对任意的 $r>0$,均有 $|G|\leqslant1$,所以隐式格式(12)绝对稳定,特别地,当 $\omega=1/72$,$r=1/6$ 时,格式(14)稳定.

为了证明三层差分格式(16)的稳定性,先引入如下两个引理.

引理 1[3] 实系数二次方程 $A\lambda^2+B\lambda+c=0$ $(A>0)$的两根按模小于等于 1 的充要条件是 $A-C\geqslant0$,$A+B+C\geqslant0$,$A-B+C\geqslant0$.

引理 2[4] 三层差分格式稳定,即传播矩阵族 $G^n(\xi)$ $(\xi=1-\cos\theta=2\sin^2\frac{\theta}{2}\overset{记}{=}2S^2\in[0,2]$,或 $S^2\in[0,1]$,$n=1,2,\cdots)$一致有界的充分必要条件是 (1) $|\lambda_{1,2}|\leqslant1$;(2) $N_0^2((1-\frac{1}{4}|G_{11}+G_{22}|^2)^2)\cap N_0^2(|(G_{11}-G_{22})^2+4G_{12}G_{21}|)\subseteq N_0^2(((G_{11}-G_{22})^2)\cap N_0^2(G_{12}^2)\cap N_0^2(G_{21}^2)$,其中 $N_0^2(f(\xi))$表示多项式 $f(\xi)$在$[0,2]$区间内所有实根的集合(重根要重复计).

注意到文[4]中 $\theta\in E$,而 $E\subset R_N$ 是 N 维欧氏空间的集合,故引理 2 对三维问题也适用.

定理 2 三层显格式(16)稳定的一个充分条件是

$$\left.\begin{aligned}&r<1/6,\quad \omega\text{ 任意},\\&-4\omega/r\leqslant\theta<1/3-4\omega/r,\theta+\varphi<(6r-1)/12r.\end{aligned}\right\}\tag{19}$$

证明 当 $\theta+\varphi\neq\frac{1}{2}$时,三层显格式(16)等价于如下的两层方程组

$$\left.\begin{aligned}\omega_{i,j,k}^{n+1}=&\frac{2}{1-2(\theta+\varphi)}\{[\frac{r}{4}(\frac{2}{3}-\theta)-\omega]\Box\omega_{i,j,k}^{n}+[\frac{r}{2}(\frac{1}{3}-\varphi)-(\frac{1}{24}-2\omega)]\Diamond\omega_{i,j,k}^{n}\\&-2(\theta+\varphi)\omega_{i,j,k}^{n}+[\frac{r}{4}\theta+\omega]\Box V_{i,j,k}^{n}\\&+[\frac{r}{2}\varphi+(\frac{1}{24}-2\omega)]\Diamond V_{i,j,k}^{n}+[\frac{1}{2}+(\theta+\varphi)]V_{i,j,k}^{n}\},\\V_{i,j,k}^{n+1}=&v_{i,j,k}^{n}.\end{aligned}\right\}$$

根据稳定性分析的分离变量法,取

$$\begin{aligned}&\omega_{i,j,k}^{n}=\omega^{n}e^{i^{*}}(p\pi xi+q\pi yj+l\pi zk),\\&V_{i,j,k}^{n}=V^{n}e^{i^{*}}(p\pi xi+q\pi yj+l\pi zk).\end{aligned}\quad(i^{*}=\sqrt{-1})\tag{20}$$

将式(20)代入式(16)并利用式(18)得

$$\begin{pmatrix}\omega^{n+1}\\V^{n+1}\end{pmatrix}=\begin{pmatrix}-B/A & -C/A\\1 & 0\end{pmatrix}\begin{pmatrix}\omega^{n}\\V^{n}\end{pmatrix},\tag{21}$$

故格式(16)的传播矩阵为

$$G(\Delta t,\beta)=\begin{pmatrix}G_{11} & G_{12}\\G_{21} & G_{22}\end{pmatrix}=\begin{pmatrix}-B/A & -C/A\\1 & 0\end{pmatrix},\tag{22}$$

其特征方程为

$$A\lambda^2+B\lambda+C=0,\tag{23}$$

其中 $G_{11}=-B/A, G_{12}=-C/A, G_{21}=1, G_{22}=0$；$A=1/2-(\theta+\varphi)$；$B=[4r-4r(\theta+\varphi)-1/3](s_1^2+s_2^2+s_3^2)-[8r/3-4r\theta-16\omega](s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)+2(\theta+\varphi)$；$C=[4r(\theta+\varphi)+1/3](s_1^2+s_2^2+s_3^2)-(4r\theta+16\omega)(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)-1/2-(\theta+\varphi)$. 则首先由条件 $A=\frac{1}{2}-(\theta+\varphi)>0$ 得

$$\theta+\varphi<\frac{1}{2},\tag{24}$$

又由条件 $A-C=1-[4r(\theta+\varphi)+1/3](s_1^2+s_2^2+s_3^2)+4(r\theta+4\omega)(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)\geqslant 0$ 对任意的 $|s_1|,|s_2|,|s_3|\in[0,1]$ 都成立的一个充分条件是 $r\theta+4\omega\geqslant 0$；$3[4r(\theta+\varphi)+1/3]\leqslant 1$. 即

$$r\theta+4\omega\geqslant 0;\quad \theta+\varphi\leqslant 0.\tag{25}$$

而条件 $A+B+C=4r(s_1^2+s_2^2+s_3^2)-\frac{8}{3}r(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)\geqslant r[4(s_1^2+s_2^2+s_3^2)-\frac{8}{3}(s_1^2+s_2^2+s_3^2)]\geqslant 0$，对任意 $|s_1|,|s_2|,|s_3|\in[0,1]$ 都成立. 最后条件 $A-B+C=-4(\theta+\varphi)+[8r(\theta+\varphi)+2/3-4r](s_1^2+s_2^2+s_3^2)-[8r\theta+32\omega-8r/3)](s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)>0$，即 $\theta+\varphi<[2r(\theta+\varphi)-r+1/6](s_1^2+s_2^2+s_3^2)+[2r/3-2r\theta-8\omega](s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)$，对任意 $|s_1|,|s_2|,|s_3|\in[0,1]$ 成立的一个充分条件是 $r\theta+4\omega<r/3$；$2r(\theta+\varphi)-r+1/6<0$；$\theta+\varphi<3[2r(\theta+\varphi)-r+1/6]$. 或 $r\theta+4\omega<r/3$；$\theta+\varphi<(6r-1)/12r$；$(1-6r)(\theta+\varphi)<1/2-3r$. 即

$$r<1/6,\quad r\theta+4\omega<r/3,\quad \theta+\varphi<(6r-1)/12r.\tag{26}$$

由此可知，$6r-1<0$，因此，$\theta+\varphi<0$，从而条件(26)强于条件(24). 故条件(24)可略去，而条件(25)，(26)可合并为

$$r<1/6, 0\leqslant r\theta+4\omega<r/3, \theta+\varphi<(6r-1)/12r.\tag{27}$$

则条件(26)成立时，条件(24)、(25)，(26)同时成立，从而引理1的条件全部满足，因此，特征方程(23)的所有根按模均为小于等于1.

其次，考虑引理2的条件(2)，因为 $G_{21}=1$，所以 $N_0^2(G_{21}^2)$是空集，故条件(2)成立的充要条件是使 $1-\frac{1}{4}G_{11}^2=G_{11}^2+4G_{12}=0$ 成立的 s_1,s_2,s_3 或者不存在或 $|s_1|,|s_2|,|s_3|\overline{\in}[0,1]$. 由 $1-G_{11}^2/4=0$ 解得 $G_{11}^2=4$. 将其代入 $G_{11}^2+4G_{12}=0$ 得 $G_{12}=-1$. 由此进一步计算得

$$-[4r(\theta+\varphi)+1/3](s_1^2+s_2^2+s_3^2)+(4r\theta+16\omega)(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)=-1.\tag{28}$$

我们分三种情况讨论(1)当 $4r(\theta+\varphi)+1/3=0$ 时，此时式(28)成为 $(r\theta+4\omega)(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)=-1/4$. 若取 $r\theta+4\omega>0$，则使该式成立的 s_1,s_2,s_3 不存在. (2)当 $4r(\theta+\varphi)+1/3<0$ 时，若取 $r\theta+4\omega>0$，则式(28)左边恒为正，而右边为-1，矛盾. 故使式(28)成立的 s_1,s_2,s_3 也不存在. (3)当 $4r(\theta+\varphi)+1/3>0$ 时，若取 $r\theta+4\omega>0$，则式(28)成为

$$[4r(\theta+\varphi)+1/3](s_1^2+s_2^2+s_3^2)=1+4(r\theta+4\omega)(s_1^2s_2^2+s_2^2s_3^2+s_3^2s_1^2)\geqslant 1.\tag{29}$$

如取 $3(4r(\theta+\varphi)+1/3)<1$，即 $\theta+\varphi<0$，则使式(28)成立的 s_1,s_2,s_3 也不存在．

综上所述，当 $r\theta+4\omega>0$ 及 $\theta+\varphi<0$ 时使关系式 $1-G_{11}^2/4=G_{11}^2+4G_{12}=0$ 成立的 s_1,s_2,s_3 并不存在．由此及条件(27)便得定理 2 所给的充分条件(19)，定理 2 证毕．

最后应指出：由于本文所构造的差分格式都满足相容性条件，由 Lax 等价性定理可推得这些格式的收敛性．数值例子表明，数值结果与理论分析相一致，因篇幅关系这里从略．

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High Accuracy Explicit Difference Schemes for Solving Three-Dimensional Equation of the Parabola

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Abstract For solving three-dimensional parabolic equation, the author proposes high accuracy explicit difference schemes of two and three strata. Of which the local truncation errors are all $O((\triangle t)^2)$, while the stability condition is $r=1/6$ and $r<1/6$ respectively.

Keywords three-dimensional equation, parabolic equation or equation of the parabola, explicit difference scheme