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对角型抛物型方程组广义解的 Hölder 连续性

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摘要 证明下面的主部是对角型的抛物型方程组广义解的整体有界性，及在抛物侧边界的 Hölder 连续性.

关键词 抛物组，对角型，广义解，整体有界性，边界 Hölder 连续性

0 引言

对角型抛物型方程组的理论是由 Ladyzenskaja 和 Uralceva 在60年代首先研究的，其结果已总结在他们和 Solonnikov 合著的书[1]中. Struwe[2]和 Giaguinta，Struwe[3]的工作还允许非线性项含有未知解的空间梯度的平方项. 文[2,3]中考虑的情形相当于下面方程(1)中取 $a^{\alpha\beta}(x,t,u,\nabla u)\equiv a^{\alpha\beta}(x,t,u)$ 的情形，借助单个线性抛物型方程的基本解[4]和填洞法，证明对角型的拟线性抛物型方程组广义解的内部 Hölder 连续性(在[2]中要求较强的条件，即 $2aM<k_0$). 现本文证明下面的抛物型方程组广义解的整体有界性及在抛物侧边界的 Hölder 连续性.

设 G 是 n 维欧氏空间的有界域，$T>0$ 是确定值，在 $Q=G\times(0,T)$ 考虑对角型抛物组

$$u_t^i-\frac{\partial}{\partial x^\alpha}\Big[a^{\alpha\beta}(x,t,u,\nabla u)\frac{\partial u^i}{\partial x^\beta}\Big]+f^i(x,t,u,\nabla u)=0\qquad(i=1,2,\cdots,N).\tag{1}$$

设 $a^{\alpha\beta}(x,t,u,\xi)$，$f^i(x,t,u,\xi)$ 分别在 $G\times(0,T)\times E^1\times E^{nN}$ 上定义，对固定的 u,ξ 关于 x,t 为可测，对固定的 x,t 关于 u,ξ 连续，且满足下面的结构条件

$$a^{\alpha\beta}(x,t,u,\nabla u)\xi^\alpha\xi^\beta\geqslant k_0|\xi|^2,\tag{2}$$

$$|a^{\alpha\beta}(x,t,u,\nabla u)|\leqslant k_1,\tag{3}$$

$$\Big(\sum_{j=1}^{N}|f^i(x,t,u,\nabla u)|^2\Big)^{1/2}\leqslant a|\nabla u|^\gamma+b,\tag{4}$$

其中常数 $k_1\geqslant k_0>0$，$a,b\geqslant 0$，$1<\gamma<2$.

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定义　称 $u=(u^1,u^2,\cdots,u^N)$ 是方程组(1)的广义解，如果 $u^i\in L^\infty(0,T,L^2(G))\cap L^2(0,T,W_2^1(G))$，当 $1<\gamma<\gamma_1=2-\frac{n}{n+2}$，$u^i\in L^\infty(0,T,L^2(G))\cap L^2(0,T,W_2^1(G))\cap L^{t^*}$，当 $\gamma_1\leqslant\gamma<2$，$\frac{n}{n+2}(1-\frac{\gamma}{2})+\frac{\gamma}{2}+\frac{1}{t^*}(\gamma-1)<1$. 并且满足

$$\int_0^t\int_G\{-u^i\varphi_t^i+\varphi_{,\alpha}^i a^{\alpha\beta}(x,t,u,\nabla u)u_{,\beta}^i+\varphi^i f^i(x,t,u,\nabla u)\}dxdt$$

$$+\int_G\varphi^i(x,t)u^i(x,t)|_{t=0}^{t=t}dx=0, \tag{1$'$}$$

$$\forall\, t\in(0,T),\varphi=(\varphi^1,\varphi^2,\cdots,\varphi^N),\varphi^i\in W_2^1(0,T,L^2(G))\cap L^2(0,T,\overset{0}{W}{}_2^1(G)),$$

其中

$$u_{,\alpha}^i=\partial u^i/\partial x^\alpha,\qquad i=1,2,\cdots,N.$$

1　主要结果

我们需要下面的引理.

引理1[1]　设 $w\in L^\infty(0,T,L^2(G))\cap L^2(0,T,\overset{0}{W}{}_2^1(G))$，那么 $w\in L^l(Q)$，$l=2(1+\frac{2}{n})$，并且存在常数 $c(n)>0$，使

$$\|w\|_{L^l(Q)}^2\leqslant c(n)\{\text{vrai}\max_{0\leqslant t\leqslant T}\int_G w^2dx+\int_0^T\int_G|\nabla w|^2\mathrm{dxdt}\}\overset{\text{记}}{=}c(n)\|w\|_Q$$

引理2[2]　设 $w\in W_1^1(B(\rho))$，在 $B(\rho)$ 的某个正测度集 S 上，$w=0$，$\zeta(x)=\zeta(|x|)$ 是 x 的取值在 $[0,1]$ 的非增连续函数，$\zeta(x)$ 在 S 上取值为1，那么存在常数 $c(n)>0$，使对 $B(\rho)$ 内任何可测子集 e，成立

$$\int_e|w(x)|\zeta(x)dx\leqslant\frac{c(n)\rho^n}{|S|}|e|^{\frac{1}{n}}\int_{B(\rho)}|\nabla w|\zeta(x)dx.$$

其中 $B(\rho)=B(x_0,\rho)=\{|x-x_0|<\rho\}$. $|S|$ 表示集合 S 的测度.

定理1　设 $1<\gamma<2$，条件(2)—(4)满足，$u=(u^1,u^2,\cdots u^N)$ 是上面定义的(1)的广义解，且存在常数 $M>0$，使

$$(|u|-M)^+=\max(|u|-M,0)\in L^2(0,T,\overset{0}{W}{}_2^1(G)),(|u|-M)^+|_{t=0}=0. \tag{5}$$

那么 $|u|=\text{vrai}\max(\sum_{i=1}^N|u^i|^2)^{1/2}$ 在 Q 整体有界.

证　设 $k\geqslant M$，用 k 取代 M，条件(5)仍成立. 为简单设 $u_t^i\in L^2(Q)$（否则下面用 φ 关于 t 的平均取代 φ^i，经过极限可导得同样结果），那么

$$\varphi^i=\begin{cases}(1-\frac{k}{|u|})u^i, & \text{当 } |u|>k,\\ 0, & \text{当 } |u|\leqslant k\end{cases}$$

可取作试验函数，代入式(1)$'$，分部积分，并利用条件(2)—(4)得

$$0=\int_0^t\int_{G\cap\{|u|>k\}}\{(1-\frac{k}{|u|})u^iu_t^i+[(1-\frac{k}{|u|})u^i]_{,\alpha}a^{\alpha\beta}(x,t,u,\nabla u)u_{,\beta}^i$$

$$+(1-\frac{k}{|u|})u^if^i(x,t,u,\nabla u)\}dxdt$$

$$\geqslant \frac{1}{2}\int_{G\cap\{|u|>k\}}(|u|-k)^2dx+\int_0^t\int_{G\cap\{|u|>k\}}\{k_0(1-\frac{k}{|u|})|\nabla u|^2+\frac{k_0k}{|u|}|\nabla(|u|)|^2$$
$$-(|u|-k)(a|\nabla u|^\gamma+b)\}dxdt,\qquad \forall t\in(0,T)$$

上式隐含了

$$\operatorname*{vrai\,max}_{0\leqslant t\leqslant T}\int_{G\cap\{|u|>k\}}(|u|-k)^2dx+\iint_{A(k)}[(1-\frac{k}{|u|})|\nabla u|^2+\frac{k}{|u|}|\nabla(|u|)|^2]dxdt$$
$$\leqslant c(k_0)\int_{A(k)}(|u|-k)(a|\nabla u|^\gamma+b)dxdt,\tag{6}$$

其中 $A(k)=Q\cap\{|u|>k\}$.

1) 先讨论$1<\gamma<\gamma_1$的情况. 用 Hölder 不等式,式(6)右边的

$$\int_{A(k)}(|u|-k)(a|\nabla u|^\gamma+b)\mathrm{dxdt}\leqslant a(\iint_{A(k)}(1-\frac{k}{|u|})|\nabla u|^2\mathrm{dxdt})^{\gamma/2}(\iint_{A(k)}|u|^ldxdt)^{1/l}|A(k)|^{1-\frac{1}{l}-\frac{\gamma}{2}}$$
$$+b(\iint_{A(k)}(|u|-k)^ldxdt)^{1/l}|A(k)|^{1-1/l}.\tag{7}$$

将式(7)代入式(6),并用 Young 不等式及引理1得

$$(\iint_{A(k)}(|u|-k)^ldxdt)^{2/l}\leqslant c\|(|u|-k)^+\|_Q$$
$$\leqslant C\{\operatorname*{vrai\,max}_{0\leqslant t\leqslant T}\int_{G\cap\{|u|>k\}}(|u|-k)^2dx+\iint_{A(k)}[(1-\frac{k}{|u|})|\nabla u|^2+\frac{k}{|u|}|\nabla(|u|)|^2]dxdt$$
$$\leqslant C\{(\iint_{A(k)}|u|^ldxdt)^{2/l(2-\gamma)}|A(k)|^{\frac{2}{2-\gamma}(1-\frac{1}{l}-\frac{\gamma}{2})}+|A(k)|^{2(1-\frac{1}{l})}$$
$$\leqslant C\{(\iint_{A(k)}[(|u|-k)^l+k^l]dxdt)^{2/l}(\iint_Q|u|^ldxdt)^{\frac{2}{l(2-\gamma)}-\frac{2}{l}}|A(k)|^\theta+|A(k)|^{2(1-1/l)}\};\tag{8}$$

其中 $\theta=\frac{2}{2-\gamma}(1-\frac{\gamma}{2}-\frac{1}{l})>0,\frac{2}{l(2-\gamma)}-\frac{2}{l}>0$. 又因 $|A(k)|\leqslant k^{-l}\iint_Q|u|^ldxdt\to0\quad(k\to\infty$,所以可取 k_0充分大. 当 $k\geqslant k_0$时,由式(8)解得

$$(\iint_{A(k)}(|u|-k)^ldxdt)^{\frac{2}{l}}\leqslant C\{k^2|A(k)|^{\frac{2}{l}+\theta}+|A(k)|^{2(1-\frac{1}{l})}\}.$$

从而

$$\iint_{A(k)}(|u|-k)dxdt\leqslant(\iint_{A(k)}(|u|-k)^ldxdt)^{\frac{1}{l}}|A(k)|^{1-\frac{1}{l}}$$
$$\leqslant C\{k|A(k)|^{\frac{1}{l}+\frac{\theta}{2}}+|A(k)|^{1-\frac{1}{l}}\}|A(k)|^{1-\frac{1}{l}}.\tag{9}$$

用 $\tau=\min(\frac{\theta}{2},1-\frac{2}{l})>0$,则由式(9)得(设 $k_0\geqslant1,|A(k)|\leqslant1$)

$$\iint_{A(k)}(|u|-k)dxdt\leqslant ck|A(k)|^{1+\tau},\tag{10}$$

其中常数 $c>0$与 k 无关. 记 $\psi(k)=\iint_{A(k)}(|u|-k)dxdt=\int_k^\infty|A(\lambda)|d\lambda$,则 $\psi'(k)=-|A(k)|$.

从而式(10)可写为

$$(C\,k)^{-1/(1+\tau)}\leqslant-[\psi(k)]^{-1/(1+\tau)}\psi'(k).\tag{11}$$

令 $k_\infty=\sup\limits_{\psi(k)>0}k$（不排除 $k_\infty=\infty$），那么 $\psi(k_\infty)=0$，$\operatorname{vrai}\max\limits_{Q}|u|\leqslant k_\infty$. 从 k_0 到 k_∞ 积分(11)，并由式(10)得

$$c^{-1/(1+\tau)}(k_\infty^{\tau/(1+\tau)}-k_0^{\tau/(1+\tau)})\leqslant[\psi(k_0)]^{\tau/(1+\tau)},$$

$$k_\infty^{\tau/(1+\tau)}\leqslant k_0^{\tau/(1+\tau)}+C^{\tau/(1+\tau)}[\psi(k_0)]^{\tau/(1+\tau)}\leqslant k_0^{\tau/(1+\tau)}(1+C|A(k_0)|^{\tau}).$$

说明了 $k_\infty<\infty$，即 $\operatorname{vrai}\max\limits_{Q}|u|<\infty$.

2) $\gamma_1\leqslant\gamma<2$ 的情况. 仅需将式(7)中的一个估计做下面的修改，而其余证明不变，即

$$\iint\limits_{A(k)}(|u|-k)|\nabla u|^{\gamma}dxdt$$

$$\leqslant\Big(\iint\limits_{A(k)}\Big(1-\frac{k}{|u|}\Big)|\nabla u|^2dxdt\Big)^{\frac{\gamma}{2}}\Big(\iint\limits_{A(k)}|u|^{l}dxdt\Big)^{\frac{2-\gamma}{l}}\Big(\iint\limits_{A(k)}|u|^{l^*}dxdt\Big)^{\frac{\gamma-1}{l^*}}|A(k)|^{1-\frac{\gamma}{2}-\frac{2-\gamma}{l}-\frac{\gamma-1}{l^*}}.$$

定理2 设 ∂G 满足条件：存在 $k_2\in(0,1)$，$R>0$，使

$$|B(x_0,\rho)\backslash G|\geqslant k_2|B(x_0,\rho)|\quad\forall\,x_0\in\partial G,\rho\leqslant R\tag{12}$$

$u=(u^1,u^2,\cdots,u^N)$，$u^i\in L^\infty(0,T,L^2(G))\cap L^2(0,T,W_2^1(G))$ 是式(1)的广义解，条件(2)—(4)满足，则存在常数 $\mu,c>0$，使

$$\max_{1\leqslant i\leqslant N}\ \operatorname*{osc}_{(B(x_0,\rho)\cap G)\times(t_0-\rho^2,t_0)}u^i\leqslant c\Big(\frac{\rho}{\rho_0}\Big)^{\mu},\forall\,x_0\in\partial G,0<t_0\leqslant T,\rho\leqslant\rho_0\leqslant\min\Big(1,\frac{R}{2},t_0^{\frac{1}{2}}\Big),\tag{13}$$

μ,c 只依赖于 n,k_0,k_1,k_2,a,b,γ 和 R（μ 不依赖 R）.

证 不妨设 $x_0=0\in\partial G$，记 $v=|u|^2=\sum\limits_{i=1}^{N}|u^i|^2$，由定理1知 $|u|$ 在 Q 整体有界，记 $M=\operatorname{vrai}\max\limits_{Q}|u|$.

1) 先证成立估计式

$$\operatorname*{vrai\,max}_{Q_1(\rho)}v\leqslant\eta\operatorname*{vrai\,max}_{Q_0(\rho)}v+2\rho(1+bM),\tag{14}$$

其中 $0<\eta<1$ 是只依赖于 $n,k_0,k_1,k_2,a,b,\gamma,M$ 的常数. $Q_0(\rho)=G(\rho)\times(t_0-\rho^2,t_0)$，$Q_1(\rho)=G\left(\frac{\rho}{2}\right)\times\left(t_0-\left(\frac{\rho}{2}\right)^2,t_0\right)$，$G(\rho)=B(x_0,\rho)\cap G$. 为此，设 $\mu_1=\operatorname*{vrai\,max}\limits_{Q_0(\rho)}v>2\rho$，否则式(14)是平凡的. 在 $Q_0(\rho)$ 上置 $w=\ln\dfrac{\mu_1}{\mu_1-u+b_1}$，$b_1=2\rho(1+bM)$. 那么 $0<b_1<\mu_1-u+b_1\leqslant\mu_1[1+(1+bM)]=\mu_1(2+bM)$. 所以在 $Q_0(\rho)$ 上

$$w\geqslant\ln\frac{1}{2+bM}.\tag{15}$$

设 $\zeta(x)=\zeta(|x|)$，$\psi(t)$ 分别为 $|x|$ 和 t 的逐段为线性的连续函数，满足：$\zeta(x)=0$，当 $|x|\geqslant\rho_0$；$\zeta(x)=1$，当 $|x|\leqslant\rho_1<\rho_0$；$\psi(t)=0$ 当 $t\leqslant t_1$；$\psi(t)=1$，当 $t\geqslant t_2>t_1$. 那么

$$|\nabla\zeta(x)|\leqslant(\rho_0-\rho_1)^{-1},0\leqslant\psi'(t)\leqslant(t_2-t_1)^{-1}.\tag{16}$$

（Ⅰ）先设 $\rho_0=\rho$，$\rho_1=\frac{3}{4}\rho$，$t_1=t_0-\rho^2$，$t_2=t_0-\frac{7}{8}\rho^2$，$u_t^i$ 存在，取 $\varphi=\zeta^2(x)\psi^2(t)u^i/(\mu_1-v+b_1)$ $(i=1,2,\cdots N)$ 代入(1)′，分部积分，利用条件(2)—(4)及 Young 不等式得

$$0=\frac{1}{2}\int_{G(\rho)}\zeta^2\psi^2wdx-\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\zeta^2\psi\psi'wdxdt$$

$$+\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\{\frac{1}{2}\zeta^2\psi^2a^{\alpha\beta}w_{,\alpha}w_{,\beta}+\zeta\zeta_{,\alpha}\psi^2a^{\alpha\beta}w_{,\beta}+\frac{\zeta^2\psi^2a^{\alpha\beta}u^i_{,\alpha}u^i_{,\beta}}{\mu_1-v+b_1}$$

$$+\frac{\zeta^2\psi^2u^if^i}{\mu_1-v+b_1}\}dxdt$$

$$\geqslant\frac{1}{2}\int_{G(\rho)}\zeta^2\psi^2wdx+\frac{k_0}{2}\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\zeta^2\psi^2|\nabla w|^2dxdt+\frac{k_0}{2}\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\frac{\zeta^2\psi^2|\nabla u|^2}{\mu_1-v+b}dxdt$$

$$-\frac{4n^2k_1}{k_0}\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\psi^2|\nabla\zeta|^2dxdt-\frac{bM}{b_1}\int_{t_0-\rho^2}^{t}\int_{G(\rho)}dxdt$$

$$-\frac{2(aM)^{2/(2-\gamma)}}{k_0b_1}\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\zeta^2\psi^2dxdt-\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\zeta^2\psi\psi'wdxdt$$

取 $t=t_0$，由上式得

$$\int_{G(\rho)}\zeta^2(x)\psi^2(t_0)w(x,t_0)dx+\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\zeta^2\psi^2|\nabla w|^2dxdt$$

$$\leqslant C\{\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)}\psi^2|\nabla\zeta|^2dxdt+\int_{t_0-\rho^2}^{t}\int_{G(\rho)}\rho^{-1}dxdt+\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)}\zeta^2\psi\psi'wdxdt\},\qquad(17)$$

$C>0$ 依赖于 n,k_0,k_1,a,b,γ 和 M. 根据式(12)和 w 的定义有(如有必要用 $v=0$ 开拓 v 到 G 外)

$$|B(\rho)\cap\{w\leqslant 0\}|\geqslant|B(\rho)\cap\{v=0\}|\geqslant k_2|B(\rho)|,\forall t\in(t_0-\rho^2,t_0).\qquad(18)$$

用引理2，式(17)右边的第三项

$$\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)}\zeta^2\psi\psi'wdxdt\leqslant\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)\cap\{w>0\}}\zeta^2\psi\psi'wdxdt$$

$$\leqslant C(n,k_2)\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)}\zeta^2\psi\psi'\rho|\nabla w|dxdt\leqslant\frac{C}{\rho}\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)\cap\{w>0\}}\zeta^2\psi|\nabla w|dxdt$$

$$\leqslant\varepsilon\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)}\zeta^2\psi^2|\nabla w|^2dxdt+\frac{1}{\varepsilon}(c\rho^{-1})^2\int_{t_0-\rho^2}^{t_0}\int_{G(\rho)}dxdt.\qquad(19)$$

联合式(17)，(19)，并适当选择 ε 得

$$\int_{G(\rho)}\zeta^2w(x,t_0)dx+\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|\nabla w|^2dxdt\leqslant C\rho^n.\qquad(20)$$

考虑到式(15)，由式(20)进一步得

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|\nabla w|^2dxdt\leqslant C\rho^n+\int_{G(\rho)}\zeta^2|w^-|dx\leqslant(C+ln\frac{1}{2+bM})\rho^n\leqslant c\rho^n,\qquad(21)$$

其中 $w^-=w-w^+$，$C>0$ 依赖于 $n,k_0,k_1,k_2,a,b,\gamma,M$.

根据(18)，再用引理2又得

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|w^+|^2dxdt\leqslant c(n,k_2)\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2\rho|\nabla(w^+)^2|dxdt$$

$$\leqslant C\rho(\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|\nabla w^+|^2dxdt)^{1/2}(\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|w^+|^2dxdt)^{1/2}.$$

从而由式(20)得

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\frac{3}{4}\rho)}|w^+|^2dxdt\leqslant\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|w^+|^2dxdt\leqslant C\rho^2\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\rho)}\zeta^2|\nabla w|^2dxdt\leqslant C\rho^{n+2}.$$

(22)

式(22)隐含了 $\lambda\to 0$时

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}|G(\frac{3}{4}\rho)\cap\{w>\lambda\}|dt\leqslant\lambda^{-2}\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\frac{3}{4})\cap\{w>\lambda\}}|w^+|^2dxdt\to 0 \tag{23}$$

并由 Lebesque 积分的绝对连续性有

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\frac{3}{4}\rho)}|(w-\lambda)^+|^2dxdt=\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\frac{3}{4}\rho)\cap\{w>\lambda\}}(w-\lambda)^2dxdt\to 0. \tag{24}$$

根据式(23),(24)对 $r\in(0,1)$取 $\lambda=\lambda(r)$,使

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}|G(\frac{3}{4}\rho)\cap\{w>\lambda\}|dt\leqslant r\rho^{n+2}, \tag{25}$$

$$\int_{t_0-\frac{7}{8}\rho^2}^{t_0}\int_{G(\frac{3}{4}\rho)}|(w-\lambda)^+|^2dxdt\leqslant r\rho^{n+2}. \tag{26}$$

(I)　设 $\zeta(x),\psi(t)$意义同前,只是现在$\frac{\rho}{2}\leqslant\rho_1<\rho_0\leqslant\frac{3}{4}\rho,t_0-\frac{7}{8}\rho^2\leqslant t_1<t_2\leqslant t_0-\frac{1}{4}\rho^2$,

取 $\varphi=\frac{\zeta^2(x)\psi^2(t)(w-k)^+}{(\mu_1-v+b_1)}u^i\quad(i=1,2,\cdots,N)$

代入式(1)′,并利用条件(2)—(4),得

$$0\geqslant\frac{1}{4}\int_{G(\rho_0)}\zeta^2\psi^2|(w-k)^+|^2dx-\frac{1}{2}\int_0^t\int_{G(\rho_0)}\zeta^2\psi\psi'|(w-k)^+|^2dxdt$$
$$+k_0\int_0^t\int_{G(\rho_0)}\frac{\zeta^2\psi^2(w-k)^+}{\mu_1-v+b_1}dxdt+\frac{k_0}{4}\int_0^t\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)^+|^2dxdt$$
$$+\frac{k_0}{2}\int_0^t\int_{G(\rho_0)}\zeta^2\psi^2(w-k)^+|\nabla(w-k)^+|^2dxdt$$
$$-2nk_1\int_0^t\int_{G(\rho_0)}\zeta|\nabla\zeta|\psi^2(w-k)^+|\nabla(w-k)^+|dxdt$$
$$-\int_0^t\int_{G(\rho_0)}\frac{\zeta^2\psi^2(w-k)^+M(a|\nabla u|^\gamma+b)}{\mu_1-v+b_1}dxdt,\forall\, t\in(0,t_0].$$

再用 Young 不等式得

$$\frac{1}{4}\int_{G(\rho_0)}\zeta^2\psi^2|(w-k)^+|^2_{t=t}dx+\frac{k_0}{4}\int_0^t\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)^+|^2dxdt$$
$$\leqslant 2n\,k_1\int_0^{t_0}\int_{G(\rho_0)}\zeta|\nabla\zeta|\psi^2(w-k)^+|\nabla(w-k)^+|dxdt$$
$$+\frac{1}{2}\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi\psi'|(w-k)^+|^2dxdt+\frac{1}{b_1}[\frac{2}{k_0}(aM)^{\frac{2}{2-\gamma}}+bM]\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2(w-k)^+dxdt$$
$$\equiv J\qquad\forall\, t\in(o,t_0].$$

对 $t\in(0,t_0]$取上确界,得

$$\frac{1}{4}\operatorname*{vrai\,max}_{0\leqslant t\leqslant t_0}\int_{G(\rho_0)}\zeta^2\psi^2|(w-k)^+|^2dx+\frac{k_0}{4}\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)^+|^2dxdt\leqslant J. \tag{27}$$

借助 Young 不等式,J 的各项

$$\int_0^{t_0}\int_{G(\rho_0)}\zeta|\nabla\zeta|\psi^2(w-k)^+|\nabla(w-k)^+|dxdt\leqslant\varepsilon\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)^+|^2dxdt$$
$$+\frac{1}{\varepsilon}\int_0^{t_0}\int_{G(\rho_0)\cap\{w>k\}}\psi^2|\nabla\zeta|^2(w-k)^2dxdt. \tag{28}$$

根据式(18)，

$|B(\rho)\cap\{w\leqslant k\}|\geqslant|B(\rho)\cap\{w\leqslant 0\}|\geqslant k_2|B(\rho)|$. 用引理2及 Young 不等式得式(29)，(30)，即

$$\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi\psi'|(w-k)^+|^2dxdt\leqslant\max\psi'\int_{t_1}^{t_0}\int_{G(\rho_0)}\zeta^2\psi|(w-k)^+|^2dxdt$$

$$\leqslant C(n,k_2)\max\psi'\int_{t_1}^{t_0}\int_{G(\rho_0)}\zeta^2\psi\rho|\nabla(w-k)^{+2}|dxdt$$

$$\leqslant\varepsilon\int_{t_1}^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)^+|^2dxdt+\frac{C\rho^2(\max\psi')^2}{\varepsilon}\int_{t_1}^{t_0}\int_{G(\rho_0)}|(w-k)^+|^2dxdt, \tag{29}$$

$$\frac{1}{b_1}\Big[\frac{2}{k_0}(aM)^{\frac{2}{2-\gamma}}+bM\Big]\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2(w-k)^+dxdt\leqslant\frac{C}{\rho}\int_{t_1}^{t_0}\int_{G(\rho_0)}\zeta^2(w-k)^+dxdt$$

$$\leqslant\frac{C}{\rho^2}\int_{t_1}^{t_0}\int_{G(\rho_0)}|(w-k)^+|^2dxdt+\int_{t_1}^{t_0}|A_{k,\rho_0}(t)|dt. \tag{30}$$

其中 $A_{k,\rho_0}(t)=G(\rho_0)\cap\{w>k\}$.

将(28)—(30)代入式(27)，并适当选择 ε 得

$$\operatorname*{vrai\,max}_{0\leqslant t\leqslant t_0}\int_{G(\rho_0)}\zeta^2\psi^2|(w-k)^+|^2dx+\int_0^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)^+|^2dxdt$$

$$\leqslant C\Big\{\Big[(\rho\max\psi')^2+\frac{1}{\rho^2}+\max|\nabla\zeta|^2\Big]\int_{t_1}^{t_0}\int_{G(\rho_0)}|(w-k)^+|^2dxdt$$

$$+\int_{t_1}^{t_0}|A_{k,\rho_0}(t)|dt\Big\}, \tag{31}$$

$C>0$依赖于 n,k_0,k_1,k_2,γ,a,b 和 M.

又考虑到 $\zeta(x),\psi(t)$ 的定义，并用引理1得

$$\int_{t_2}^{t_0}\int_{G(\rho_1)}|(w-k)^+|^2dxdt\leqslant\int_{t_1}^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2|(w-k)^+|^2dxdt$$

$$\leqslant\Big(\int_{t_1}^{t_0}\int_{G(\rho_0)}|\zeta\psi(w-k)^+|^ldxdt\Big)^{2/l}\Big(\int_{t_1}^{t_0}\int_{G(\rho_0)\cap\{w>k\}}dxdt\Big)^{\frac{2}{n+2}}$$

$$\leqslant C\Big\{\operatorname*{vrai\,max}_{t_1\leqslant t\leqslant t_0}\int_{G(\rho_0)}|\zeta(\psi(w-k)^+|^2dx+\int_{t_1}^{t_0}\int_{G(\rho_0)}|\nabla(\zeta\psi(w-k)^+)|^2dxdt\Big\}\Big(\int_{t_1}^{t_0}|A_{k,\rho_0}(t)|dt\Big)^{\frac{2}{n+2}}$$

$$\leqslant C\Big\{\operatorname*{vrai\,max}_{t_1\leqslant t\leqslant t_0}\int_{G(\rho_0)}|(\zeta\psi(w-k)^+|^2dx+\int_{t_1}^{t_0}\int_{G(\rho_0)}\zeta^2\psi^2|\nabla(w-k)|^2dxdt$$

$$+\int_{t_1}^{t_0}\int_{G(\rho_0)}|\nabla\zeta|^2|(w-k)^+|^2dxdt\Big\}\Big(\int_{t_1}^{t_0}|A_{k,\rho_0}(t)|dt\Big)^{\frac{2}{n+2}}. \tag{32}$$

联合式(31)，(32)，(16)和引理1，可见对任何 $h>k\geqslant 0$ 成立

$$(h-k)^2\Big(\int_{t_2}^{t_0}|A_{k,\rho_1}(t)|dt\Big)\leqslant\int_{t_2}^{t_0}\int_{G(\rho_1)}|(w-k)^+|^2dxdt$$

$$\leqslant C\Big\{\Big[\frac{\rho^2}{t_2-t_1}+\frac{1}{\rho^2}+\frac{1}{(\rho_0-\rho_1)^2}\Big]\int_{t_1}^{t_0}\int_{G(\rho_0)}|(w-k)^+|^2dxdt$$

$$+\Big(\int_{t_1}^{t_0}|A_{k,\rho_0}(t)|dt\Big)\Big\}\Big(\int_{t_1}^{t_0}|A_{k,\rho_0}(t)|dt\Big)^{\frac{2}{n+2}}. \tag{33}$$

$C>0$依赖于$n,k_0,k_01,k_2,\gamma,a,b$ 和 M，与 $\rho_0,\rho_1,\rho,h,k,t_1$无关. 对 $v=0,1,2,\cdots$，置 $\rho_v=\frac{\rho}{2}+\frac{1}{2^v}\cdot\frac{\rho}{4}$，$t_{v+1}=t_0-(\frac{\rho^2}{4}+\frac{1}{2^v}\cdot\frac{5\rho^2}{8})$，$k_v=2\lambda-\frac{\lambda}{2^v}$，

$$A_v=(\int_{t_{v+1}}^{t_0}|A_{k_v,\rho_v}(t)|dt),I_v=\int_{t_{v+1}}^{t_0}\int_{G(\rho_v)}|(w-k)^+|^2dxdt.$$

分别用 ρ_v,ρ_{v+1}取代 ρ_0,ρ_1，用 k_{v+1},k_v 取代 h,k，用 t_{v+1}取代 t_1，由式(33)给出

$$(\frac{\lambda}{2^{v+1}})^2A_{v+1}\leqslant C\{(\frac{8.\,2^{v+1}}{5\rho^2}+\frac{(4.\,2^{v+1})^2}{\rho^2}+\frac{1}{\rho^2})I_v+A_v\}A_v^{\frac{2}{n+2}},\tag{34}$$

$$I_{v+2}\leqslant C\{(\frac{8.\,2^{v+1}}{5\rho^2}+\frac{(4.\,2^{v+1})^2}{\rho^2}+\frac{1}{\rho^2})I_v+A_v\}A_v^{\frac{2}{n+2}}.\tag{35}$$

因式(25)，(26)表明，$v=0$时

$$A_0\leqslant r\rho^{n+2},I_0\leqslant r\rho^{n+2}.\tag{36}$$

假设 $v>0$时，成立

$$A_v\leqslant r\delta^v\rho^{n+2},I_v\leqslant r\delta^v\rho^{n+2}.\tag{37}$$

那么由式(34)，(35)继续得

$$A_{v+1}\leqslant\frac{2^9cr^{\frac{2}{n+2}}}{\lambda^2}(4^2\delta^{\frac{2}{n+2}})^v(r\delta^v\rho^{n+2}),\tag{38}$$

$$I_{v+1}\leqslant 2^9cr^{\frac{2}{n+2}}(4^2\delta^{\frac{2}{n+2}})^v(r\delta^v\rho^{n+2}).\tag{39}$$

如果取 $\delta>0,r\in(0,1)$，使满足$4^2\delta^{\frac{2}{n+2}}=1,2^9cr^{\frac{2}{n+2}}\leqslant\delta$，并取 $\lambda\geqslant\lambda(r)+1$（$\lambda(r)$为使式(25)，(26)成立的常数)，那么由式(38)，(39)立见式(37)继续对 $v+1$成立，根据归纳法式(37)对一切正整数 v 成立. 因此

$$\iint_{Q_1(\rho)\cap\{w>2\lambda\}}dxdt=\int_{t_0-\frac{\rho^2}{4}}^{t^0}|A_{2\lambda},\frac{\rho}{2}(t)|dt=\lim_{v\to\infty}A_v=0.$$

这隐含了$\operatorname{vrai}\max\limits_{Q_1(\rho)}w\leqslant2\lambda$，即$\operatorname{vrai}\max\limits_{Q_1(\rho)}u\leqslant(1-e^{-2\lambda})\mu_1+b_1$.

于是式(14)得证($\eta=1-e^{-2\lambda}$).

2）用(14)式进行迭代. 对任意 $\rho\leqslant\rho_0=\min(1,\frac{R}{2},t_0^{\frac{1}{2}})$，可找到 $v\geqslant0$，使

$$\rho_{v+1}=\frac{\rho_0}{2^{v+1}}<\rho\leqslant\frac{\rho_0}{2^v}=\rho_v.\tag{40}$$

由式(14)得

$$\operatorname*{vrai\,max}_{G(\rho_v)\times(t_0-\rho_v^2,t_0)}v=\operatorname*{vrai\,max}_{G(\rho_{v-1}/2)\times(t_0-(\frac{\rho_{v-1}}{2})^2,t_0)}v\leqslant\eta\operatorname*{vrai\,max}_{G(\rho_{v-1})\times(t_0-\rho_{v-1}^2,t_0)}v+2\rho_{v-1}(1+bM).$$

因放大 η，式(14)保持成立，因此可认为$2\eta>1$. 经过迭代，并根据式(40)得

$$\operatorname*{vrai\,max}_{G(\rho)\times(t_0-\rho^2,t_0)}v\leqslant\operatorname*{vrai\,max}_{G(\rho_v)\times(t_0-\rho_v,t_0)}v\leqslant\eta^v\operatorname*{vrai\,max}_{G(\rho_0)\times(t_0-\rho_0^2,t_0)}v+2\rho_0(1+bM)\eta^{v-1}[1+\frac{1}{2\eta}+\cdots+(\frac{1}{2\eta})^{v-1}]$$

$$\leqslant\eta^v[\operatorname*{vrai\,max}_{G(\rho_0)\times(t_0-\rho_0^2,t_0)}v+\frac{4\rho_0(1+bM)}{2\eta-1}]\leqslant\eta^v[M^2+\frac{4\rho_0(1+bM)}{2\eta-1}]$$

$$=C\eta^v=\frac{C}{\eta}(\frac{1}{2^{v+1}})\log_2\frac{1}{\eta}\leqslant\frac{c}{\eta}(\frac{\rho}{\rho_0})\log_2\frac{1}{\eta}$$

$$\leqslant C(\frac{\rho}{\rho_0})^{2\mu}, 2\mu = \log_2 \frac{1}{\eta} > 0.$$

上式隐含了定理2的结论(13)成立.

以上证明中设 u_t^i 存在. 但只要象文[1]或文[4]那样,通过极限过程,可不需设 u_t^i 存在,对定理中的函数 u^i,同样有式(17),(27)成立. 从而定理仍正确. 从定理2的证明可见,成立.

定理3　设条件(2),(3),(12)满足,如果 $(\sum_{i=1}^{N}|f^i(x,t,u,\nabla u)|^2)^{1/2} \leqslant a|\nabla u|^2 + b, u^i \in L^2(0,T,W_2^1(G)) \in L^\infty(Q)$ 满足式(1)′,及 $aM \leqslant k_0, M = |u|_{L^\infty(Q)}$,那么定理2的结论仍成立.

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Hölder Continuity of Generalized Solution for the Parabolic Equations of Diagonal Form

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Abstract　The author demonstrates the global boundedness of generalized solution for the parabolic equations with diagonal principal part, and the Hölder continuity near parabolic lateral boundary.

Key words　parabolic system, diagonal form, generalized solution, boundary Hölder Continuity, global bonudedness