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椭圆抛物型偏微分奇摄动混合边值问题数值解*

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摘要 本文对椭圆抛物型偏微分方程奇异摄动混合边值问题构造一种差分格式，并在较弱的条件下证明这个格式的一阶一致收敛性。

关键词 奇异摄动问题，椭圆抛物型方程，混合边值问题，一致收敛性.

0 引言

1980年，苏煜城、吴启光在文〔1〕中首先讨论椭圆抛物型偏微分方程第一边值问题：

$$\begin{cases} \angle u \equiv \varepsilon \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2} - a(x,y)\dfrac{\partial u}{\partial y} = f(x,y) \qquad x,y\in(0,1)\times(0,1), \\ u|_\Gamma = 0, \end{cases}$$

($a(x,y)\geqslant a>0$) 的$Il'in$格式的收敛性和解的渐近性态。1984年，林鹏程、刘发旺在文〔2〕中证明$Il'in$格式是一阶一致收敛性。1985年，苏煜城在文〔3〕从另外的途径证明下列问题的一阶一致收敛性。

$$\begin{cases} \angle u \equiv \varepsilon \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2} - a(x,y)\dfrac{\partial u}{\partial y} + c(x,y)u = f(x,y), \qquad x,y\in(0,1)\times(0,1), \\ u|_\Gamma = 0, \end{cases}$$

其中系数满足一定条件。

对于椭圆抛物型奇摄动混合边值问题：

$$\begin{cases} \angle u \equiv \varepsilon \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial x^2} - a(x,y)\dfrac{\partial u}{\partial y} = f(x,y), \qquad x,y\in(0,1)\times(0,1), \\ \left[\alpha u - \beta\dfrac{\partial u}{\partial x}\right]\Big|_{x=0} = \varphi_1(y), \qquad \left[\gamma u + \zeta\dfrac{\partial u}{\partial x}\right]\Big|_{x=1} = \varphi_2(y), \\ u|_{y=0} = \Psi_1(x), \qquad u|_{y=1} = \Psi_2(x), \end{cases}$$

刘发旺在福州大学硕士毕业论文中讨论上述问题差分格式的收敛性和解的渐近性态。本文对

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上述问题构造一种差分格式，在较弱的条件下，证明了一阶一致收敛性。

1 微分方程的一些性质

设$\Omega=\{(x,y)|0<x,y<1\}$，$\bar{\Omega}=\{(x,y|0\leqslant x,y\leqslant 1\}$，在$\bar{\Omega}$上考虑如下问题：

$$\angle u\equiv\varepsilon\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}-a(x,y)\frac{\partial u}{\partial y}=f(x,y),\qquad (x,y)\in\Omega,\tag{1}$$

$$l_1u(0,y)\equiv\left[\alpha u-\beta\frac{\partial u}{\partial x}\right]\Big|_{x=0}=\varphi_1(y),\qquad 0\leqslant y\leqslant 1,\tag{2}$$

$$l_2u(1,y)\equiv\left[\gamma u+\delta\frac{\partial u}{\partial x}\right]\Big|_{x=1}=\varphi_2(y),\qquad 0\leqslant y\leqslant 1,\tag{3}$$

$$l_3u(x,0)\equiv u(x,0)=\Psi_1(x),\qquad 0\leqslant x\leqslant 1,\tag{4}$$

$$l_4u(x,1)\equiv u(x,1)=\Psi_2(x),\qquad 0\leqslant x\leqslant 1.\tag{5}$$

其中$A>a(x,y)>a>0$，$a(x,y)$ $f(x,y)$，$\varphi_1(y)$，$\varphi_2(y)$，$\Psi_1(x)$，$\Psi_2(x)$为充分光滑的函数，$\alpha,\beta,\gamma,\delta\geqslant 0$，$\alpha+\beta>0$，$\gamma-\delta>0$，且满足下列相容性条件，即

1° $\alpha\Psi_2(0)-\beta\Psi_2'(0)=\varphi_1(1)$，　$\alpha\Psi_1(0)-\beta\Psi_1'(0)=\varphi_1(0)$，

$\gamma\Psi_2(1)+\delta\Psi_2'(1)=\varphi_2(1)$，　$\gamma\Psi_1(1)+\delta\Psi_1'(1)=\varphi_2(0)$。

2° 在正方形区域$\bar{\Omega}$的四个角点T

$$\frac{\partial a(x,y)}{\partial x}\Big|_T=0.$$

3° $[\alpha f-\beta f_x]_{x=0,y=0}=\varepsilon\varphi_1''(0)+\alpha\Psi_1''(0)-\beta\Psi_1'''(0)-a(0,0)\varphi_1'(0)$，

$[\alpha f-\beta f_x]_{x=0,y=1}=\varepsilon\varphi_1''(1)+\alpha\Psi_2''(0)-\beta\varphi_2'''(0)-a(0,1)\varphi_1'(1)$，

$[\alpha f-\beta f_x]_{x=1,y=0}=\varphi_2''(0)+\gamma\Psi_1''(1)+\delta\Psi_1'''(1)-a(1,0)\varphi_2'(0)$，

$[\alpha f-\beta f_x]_{x=1,y=1}=\varphi_2''(1)+\gamma\Psi_2''(1)+\delta\Psi_2'''(1)-a(1,1)\varphi_2'(1)$。

下面讨论微分方程（1）—（5）的一些性质。容易证明如下二个引理：

引理1 设$u(x,y)$是$\bar{\Omega}$区域上非恒贫常数的光滑函数，且满足

$$\angle u\leqslant 0,\qquad\qquad (x,y)\in\Omega,$$
$$l_1u(0,y)\geqslant 0,\quad l_2u(1,y)\geqslant 0,\quad 0\leqslant y\leqslant 1,$$
$$l_3u(x,0)\geqslant 0,\quad l_4u(x,1)\geqslant 0,\quad 0\leqslant x\leqslant 1,$$

则

$$u(x,y)\geqslant 0,\qquad (x,y)\in\bar{\Omega},$$

引理2 若$|\angle u(x,y)|\leqslant k\{1+\varepsilon^{-1}exp-\frac{a(1-y)}{2\varepsilon})\}$，$|l_1u(0,y)|\leqslant k_1$，$|l_2u(1,y)|\leqslant k_2$，$|l_3u(x,1)|\leqslant k_3$，$|l_4u(x,1)|\leqslant k_4$，则$|u(x,y)|\leqslant c$。

引理3 问题（1）—（5）的解可表示为

$$u(x,y)=W(x,y)+V(x,y)+R(x,y),$$

其中$W(x,y)$满足

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$$\begin{cases} \dfrac{\partial^2 w}{\partial x^2} - a(x,y)\dfrac{\partial w}{\partial y} = f(x,y), \\ \left[\alpha w - \beta\dfrac{\partial w}{\partial x}\right]\Big|_{x=0} = \varphi_1(y), \qquad \left[\gamma w + \delta\dfrac{\partial w}{\partial x}\right]\Big|_{x=1} = \varphi_2(y), \\ W|_{y=0} = \Psi_1(x), \end{cases}$$

$$V(x,y) = \Phi(x)\exp\left(-\frac{a(x,1)(1-y)}{\varepsilon}\right) \equiv \left[-w(x,1)+\Psi_2(x)\right]\exp\left(-\frac{a(x,1)(1-y)}{\varepsilon}\right),$$

$$|R(x,y)| \leqslant c\varepsilon,$$

证明　令$R(x,y)=U(x,y)-W(x,y)-V(x,y)$由相容性条件推出：$\alpha\Phi(0)-\beta\Phi'(0)=0$，$\gamma\Phi(1)+\delta\Phi'(1)=0$，所以

$$\begin{aligned} l_1R(0,y) &= \varphi_1(y)-\varphi_1(y)-l_1v(0,y) \\ &= -\left\{\alpha\Phi(0)\exp\left(-\frac{a(0,1)(1-y)}{\varepsilon}\right)-\beta\Phi'(0)\exp\left(-\frac{a(0,1)(1-y)}{\varepsilon}\right)\right. \\ &\quad \left.-\beta\Phi(0)\left(-\frac{\partial a(0,1)}{\partial x}\cdot\frac{1-y}{\varepsilon}\right)\cdot\exp\left(-\frac{a(0,1)(1-y)}{\varepsilon}\right)\right\}, \end{aligned}$$

$$|l_1R(0,y)| \leqslant c\varepsilon.$$

同理可证

$$|l_2R(1,y)| \leqslant c\varepsilon.$$

所以

$$l_3R(x,0) = \Psi_1(x)-\Psi_1(x)-v(x,0) = -\Phi(x)\exp\left(-\frac{a(x,1)}{\varepsilon}\right),$$

$$|l_3R(x,0)| \leqslant c\cdot\varepsilon,$$

$$l_4R(x,1) = u(x,1)-w(x,1)-v(x,1) = \Psi_2(x)-w(x,1)-\Phi(x) = 0,$$

又因为

$$|\angle R(x,y)| \leqslant c\cdot\varepsilon\left\{1+\varepsilon^{-1}\exp\left(-\frac{a(1-y)}{\varepsilon}\right)\right\},$$

根据引理2推得

$$|R(x,y)| \leqslant c\varepsilon.$$

类似于文〔3〕、〔4〕的讨论，容易证明：

引理4　若$u(x,y)$是式(1)—(5)的解，则$\left|\dfrac{\partial^i u}{\partial x^i}\right| \leqslant M$，$\left|\dfrac{\partial^i u}{\partial y^i}\right| \leqslant M\left\{1+\varepsilon^{-i}\exp\left(-\dfrac{a(1-y)}{\varepsilon}\right)\right\}$.

这引理的结讨可进一步推广.

引理5　若$u(x,y)$是式(1)—(5)的解，则$\left|\dfrac{\partial^{i+j}u}{\partial x^i\partial y^j}\right| \leqslant M\left\{1+\varepsilon^{-j}\exp\left(-\dfrac{a(1-y)}{\varepsilon}\right)\right\}$.

容易把解的奇异性分离出来：

引理6　式(1)—(5)的解可表示为

$$u(x,y) = r(x)\tilde{V}(x,y)+Z(x,y),$$

其中

$$\tilde{V}(x,y) = \exp\left(-\frac{a(x,1)(1-y)}{\varepsilon}\right), \qquad |r(x)| \leqslant M,$$

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$$\left|\frac{\partial^i z}{\partial x^i}\right| \leqslant M, \qquad \left|\frac{\partial^j z}{\partial y^j}\right| \leqslant M\left\{1+\varepsilon^{-j+1}\exp\left(-\frac{a(1-y)}{\varepsilon}\right)\right\}.$$

证明 取$r(x)=\varepsilon\dfrac{\partial u(x,1)}{\partial y}/a(x,1)$，则 $|r(x)|\leqslant M$，令 $z(x,y)=u(x,y)-r(x)\tilde{V}(x,y)$，直接计算推出$\left|\dfrac{\partial^i z}{\partial x^i}\right|\leqslant M$。定义新算子$L_1$

$$L_1u(x,y)\equiv\varepsilon\frac{\partial^2 u}{\partial y^2}-a(x,y)\frac{\partial u}{\partial y}$$

$$\begin{aligned}L_1\frac{\partial z}{\partial y}&=\varepsilon\frac{\partial^3 z}{\partial y^3}-a(x,y)\frac{\partial^2 z}{\partial y^2}\\&=\varepsilon\frac{\partial^3 u}{\partial y^3}-a(x,y)\frac{\partial^2 u}{\partial y^2}-r(x)\left[\varepsilon\frac{\partial^3\tilde{V}}{\partial y^3}-a(x,y)\frac{\partial^2\tilde{V}}{\partial y^2}\right]\\&=\frac{\partial f}{\partial y}-\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial a}{\partial y}\frac{\partial u}{\partial y}-r(x)\cdot[a(x,1)-a(x,y)]\frac{\partial^2\tilde{V}}{\partial y^2}\\&\stackrel{\triangle}{=}G(x,y).\end{aligned}$$

由引理5得

$$\left|\frac{\partial^j G}{\partial y^j}\right|\leqslant M\left\{1+\varepsilon^{-j-1}\exp\left(-\frac{a(1-y)}{\varepsilon}\right)\right\},$$

$$\left|\frac{\partial z}{\partial y}\Big|_{y=0}\right|\leqslant M, \qquad \frac{\partial z}{\partial y}\Big|_{y=1}=0,$$

由文〔5〕引理2.3推出

$$\left|\frac{\partial^j z}{\partial y^j}\right|\leqslant M\left\{1+\varepsilon^{-j+1}\exp\left(-\frac{a(1-y)}{\varepsilon}\right)\right\}.$$

2 差分格式及其一致收敛性

对$\bar{\Omega}$进行网格划分：

$$x_i=ih,\quad i=0,1,\cdots,N,\quad Nh=1,$$
$$y_j=j\tau,\quad j=0,1,\cdots,M,\quad M\tau=1,$$

对问题(1)—(5)，构造下列差分格式

$$L^{h\tau}u_{ij}\equiv\varepsilon\sigma_{ij}\delta_y^2u_{ij}+\delta_x^2u_{ij}-a_{ij}D_{0y}u_{ij}=f_{ij},$$
$$1\leqslant 1\leqslant N-1,\qquad 1\leqslant j\leqslant M-1, \tag{6}$$
$$l_1^{h\tau}u_{oj}\equiv\alpha u_{oj}-\beta(u_{1j}-u_{0j})/h=\varphi_{1j},\qquad 1\leqslant j\leqslant M-1, \tag{7}$$
$$l_2^{h\tau}u_{Nj}\equiv\gamma u_{Nj}+\delta(u_{Nj}-u_{N-1,j})/h=\varphi_{2j}\qquad 1\leqslant j\leqslant M-1, \tag{8}$$
$$l_3^{h\tau}u_{io}\equiv u_{io}=\Psi_{1i},\qquad 0\leqslant i\leqslant N, \tag{9}$$
$$l_4^{h\tau}u_{iM}\equiv u_{iM}=\Psi_{2i},\qquad 0\leqslant i\leqslant N, \tag{10}$$

其中

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$$\sigma_{ij}=\frac{a_{ij}\rho}{2}\coth\frac{a_{ij}\rho}{2},\qquad \rho=\tau/\varepsilon,\qquad a_{ij}=a(x_i,y_j),\qquad f_{ij}=f(x_i,y_j),$$
$$\varphi_{1j}=\varphi_1(y_j),\qquad \varphi_{2j}=\varphi_2(y_j),\qquad \Psi_{1i}=\Psi_1(x_i),\qquad \Psi_{2i}=\Psi_2(x_i)$$

如下面极值原理成立。

引理7　若$L^{h\tau}u_{ij}\leqslant 0$，$1\leqslant i\leqslant N-1$，$1\leqslant j\leqslant M-1$，$l_1^{h\tau}u_{0j}\geqslant 0$，$l_2^{h\tau}u_{Nj}\geqslant 0$，$1\leqslant j\leqslant M-1$，$l_3^{h\tau}u_{i0}\geqslant 0$，$l_4^{h\tau}u_{iM}\geqslant 0$，$0\leqslant i\leqslant N$，则对任何$i$和$j$均有$u_{ij}\geqslant 0$。

证明　把式(7)—(10)改写为

$$-l_1^{h\tau}u_{oj}=-\varphi_{1j},\qquad l_2^{h\tau}u_{Nj}=-\varphi_{2j},\qquad -l_3^{h\tau}u_{io}=-\Psi_{1i};$$
$$-l_4^{h\tau}u_{iM}=-\Psi_{2i},$$

和式(6)一起改写为矩阵形式：$AU=F$

其中

$$U=[u_0,\ u_1,\ \cdots,\ u_M]^T,\qquad F=[F_0,\ F_1,\ \cdots,\ F_M]^T,$$
$$U_j=[u_{0j},\ u_{1j},\ \cdots,\ u_{Nj}]^T,\qquad 0\leqslant j\leqslant M,$$
$$F_0=-[-\Psi_{10},\ -\Psi_{11},\ \cdots,\ -\Psi_{1N}]^T,$$
$$F_j=-[-\varphi_{1j},\ f_{1j},\ \cdots,\ f_{N-1,j},\ -\varphi_{2j}]^T,\qquad 1\leqslant j\leqslant M-1,$$
$$F_M=[-\Psi_{20},\ -\Psi_{21},\ \cdots,\ -\Psi_{2N}]^T,$$

A为$(M+1)\times(M+1)$块三对角矩阵，每一块矩阵为$(N+1)\times(N+1)$矩阵，易见A的对角线元素为(-1)，$\left(-\frac{2\varepsilon\sigma_{ji}}{\tau^2}-\frac{2}{h^2}\right)$，$\left(-a-\frac{\beta}{h}\right)$，$\left(-r-\frac{\delta}{h}\right)$都是负，$A$的非零非对角线元素为$\frac{1}{h^2}$，$\left(\frac{\varepsilon\sigma_{ij}}{\tau^2}+\frac{a_{ij}}{2\tau}\right)$，$\left(\frac{\varepsilon\sigma_{ij}}{\tau^2}-\frac{a_{ij}}{2\tau}\right)$，$\frac{\beta}{h}$，$\frac{\delta}{h}$均非负，且$A$的同一行元素之和非正，故$A$为不可约的$M$阵，$A^{-1}\leqslant 0$，因此$U\geqslant 0$。

引理8　若

$$\begin{cases}\left|L^{h\tau}u_{ij}\right|\leqslant K\left\{1+\frac{1}{\max(\tau,\varepsilon)}\exp\left(-\frac{a(1-y_{j+1})}{\varepsilon}\right)\right\},\\ |l_1^{h\tau}u_{oj}|\leqslant k_1,\qquad |l_2^{h\tau}u_{Nj}|\leqslant k_2,\qquad |l_3^{h\tau}u_{io}|\leqslant k_3,\qquad |l_4^{h\tau}u_{iM}|\leqslant k_4,\end{cases}$$

则$|u_{ij}|\leqslant C$。

证明　(1)当$r+a>0$时，不妨设$\triangle=r(a+\beta)+a\delta=1$。取闸函数

$$\Phi(x_i,y_j)=\varphi_0(y_j)+\varphi_1(x_i,y_j)\pm u_{ij},$$

其中

$$\varphi_0(y)=r_0\exp\left(-\frac{a[1-(y+\tau)]}{\varepsilon}\right),$$
$$\varphi_1(x,y)=k_1p_1(x)+k_2p_2(x)+k_3p_3(y)+(k_4+r_1)p_4(y),$$

这里　$p_1(x)=r(1-x)+\delta$，$p_2(x)=ax+\beta$，$p_3(y)=1-y$，$p_4(y)=y$，

$$L^{h\tau}\varphi_0(y)=\varepsilon\sigma_{ij}\frac{\varphi_0(y)}{\tau^2}[\exp(a\rho)+\exp(-a\rho)-2]-\frac{a_{ij}}{2\tau}\varphi_0(y)$$
$$[\exp(a\rho)-\exp(-a\rho)]$$
$$\leqslant-\frac{a_{ij}}{\tau}\varphi_0(y)\operatorname{sh}\frac{a\rho}{2}\operatorname{sh}\frac{(a_{ij}-a)\rho}{2}\operatorname{sh}^{-1}\frac{a_{ij}\rho}{2}$$

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$$\leqslant C_0 \frac{1}{\max(\tau,\varepsilon)} \exp\left(-\frac{a[1-(y+\tau)]}{\varepsilon}\right),$$

C_0为某一固定常数。

$$L^{h\tau}p_1(x)=L^{h\tau}p_2(x)=0, \qquad L^{h\tau}p_3(y)=a_{ij}, \qquad L^{h\tau}p_4(y)=-a_{ij},$$

当取$r_0=k/C_0$，$r_1\geqslant k_3-k_4+k/a$的任意正数，可推出$L^{h\tau}\Phi(x_i,y_j)\leqslant 0$，因为$l_1^{h\tau}\Phi(0,y_j)\geqslant 0$，$l_2^{h\tau}\Phi(1,y_j)\geqslant 0$，$l_3^{h\tau}\Phi(x_i,0)\geqslant 0$，$l_4^{h\tau}\Phi(x_i,1)\geqslant 0$，根据引理7得 $|u_{ij}|\leqslant C$。

(2)当$r=a=0$时，不妨设$\beta=\delta=1$，取闸函数与情形(1)一样，只不过这里

$$p_1(x)=\frac{x(x+h-2)}{2(1-h)}+1, \qquad p_2(x)=\frac{x(x-h)}{2(1-h)}, \qquad r_0=\frac{k}{c_0},$$

$r_1\geqslant k_3-k_4+[k+2(k_1+k_2)]/a$的任意正常数。

易证$\Phi(x_i,y_j)$满足引理7的条件，所以$|u_{ij}|\leqslant C$。 证毕。

定义

$$L^{h\tau}[r(x_i)v_{ij}]=L[r(x_i)v(x_i,y_j)],$$

$$l_1^{h\tau}[r_0v_{0j}]=l_1[r(0)v(0,y_j)], \qquad l_2^{h\tau}[r_Nv_{Nj}]=l_2[r(1)v(1,y_j)], \tag{11}$$

$$l_3^{h\tau}[r_iv_{i0}]=l_3[r(x_i)v(x_i,0)], \qquad l_4^{h\tau}[r_iv_{iM}]=l_4[r(x_i)v(x_i,1)],$$

和

$$\begin{cases} L^{h\tau}z_{ij}=Lz(x_i,y_j), \\ l_1^{h\tau}z_{0j}=l_1z(0,y_j), \qquad l_2^{h\tau}z_{Nj}=l_2z(1,y_j), \\ l_3^{h\tau}z_{i0}=l_3z(x_i,0), \qquad l_4^{h\tau}z_{iM}=l_4z(x_i,1), \end{cases} \tag{12}$$

则差分方程(6)—(10)的解 $u_{ij}=r(x_i)v_{ij}+z_{ij}$。

引理9 若$r(x_i)v_{ij}$是方程(11)的解，则$|r(x_i)[v(x_i,y_j)-v_{ij}]|\leqslant C\cdot(h+\tau)$。

证明

$$L[r(x)v(x,y)]=\varepsilon r(x)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2}{\partial x^2}[r(x)v(x,y)]-a(x,y)r(x)\frac{\partial v}{\partial y}$$

$$=r(x)\cdot\frac{a(x,1)[a(x,1)-a(x,y)]}{\varepsilon}v(x,y)+\frac{\partial^2}{\partial x^2}[r(x)v(x,y)], \tag{13}$$

$$L^{h\tau}[r(x_i)v(x_i,y_j)]=\varepsilon\sigma\delta_y^2[r(x_i)v(x_i,y_j)]+\delta_x^2[r(x_i)v(x_i,y_j)] - a(x_i,y_j)D_{0y}[r_iv(x_i,y_j)]$$

$$=\frac{2a(x_i,y_j)}{\tau}\cdot\frac{\operatorname{sh}\frac{a(x_i,1)\rho}{2}\cdot\operatorname{sh}\frac{a(x_i,1)-a(x_i,y_j)\rho}{2}}{\operatorname{sh}\frac{a(x_i,y_j)\rho}{2}}r_iv(x_i,y_j)+\delta_x^2[r_iv(x_i,y_j)],$$

根据文[5]，易证存在C_1，使得当$\tau\leqslant C_1$时，

$$|L^{h\tau}[r(x_i)(v(x_i,y_j)-v_{ij})]|=|L[r(x_i)v(x_i,y_j)]-L^{h\tau}[r_iv_{ij}]|$$

$$\leqslant\frac{C\tau^2(1-y_j)}{\varepsilon^2(\tau+\varepsilon)}\exp\left(-\frac{a(1-y_j)}{\varepsilon}\right)+ch^2\leqslant C\frac{\tau}{\max(\tau,\varepsilon)}\exp\left(-\frac{a(1-y_{j+1})}{\varepsilon}\right)+ch^2,$$

$$l_1^{h\tau}[r_0v_{0j}-r(0)v(0,y_j)]$$

$$=l_1[r(0)v(0,y_j)]-l_1^{h\tau}[r(0)v(0,y_j)]$$

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$$= -\beta \frac{\partial [r(x)v(x,y)]}{\partial x}\Big|_{x=0} + \beta \frac{r(h)v(h,y_j) - r(0)v(0,y_j)}{h}$$

$$= -\beta[r'(0)v(0,y_j) + r(0)\frac{\partial v(0,y_j)}{\partial x}] + \beta\frac{[r(h)-r(0)]v(h,y_j) + r(0)[v(h,y_j) - v(0,y_j)]}{h}$$

$$= -\beta[r'(0)v(0,y_j) + r(0)\frac{\partial v(0,y_j)}{\partial x}] + \beta\{[r'(0) + \frac{h^2}{2}r''(\xi_1)]v(h,y_j)$$

$$+ r(0)[\frac{\partial v(0,y_j)}{\partial x} + \frac{h}{2}\frac{\partial^2 v(\xi_2,y_j)}{\partial x^2}]\}$$

$$= \beta r'(0)[v(h,y_j) - v(0,y_j)] + \beta\cdot\frac{h}{2}r''(\xi_1)v(h,y_j) + r(0)\frac{h}{2}\frac{\partial^2 v(\xi_2,y_j)}{\partial x^2}$$

$$= \beta r'(0)\cdot h\frac{\partial v(\xi_3,y_j)}{\partial x} + \beta\cdot\frac{h}{2}r''(\xi_1)v(h,y_j) + r(0)\cdot\frac{h}{2}\frac{\partial^2 v(\xi_2,y_j)}{\partial x^2},$$

其中 $0<\xi_1$，ξ_2，$\xi_3<h$，所以

$$|l_1^{h\tau}[r_0 v_{0j} - r(0)v(0,y_j)]| \leqslant Ch.$$

同理可证

$$|l_2^{h\tau}[r_N v_{Nj} - r(1)v(1,y_j)]| \leqslant Ch,$$

$$l_3^{h\tau}[r_i v_{i0} - r(x_i)v(x_i,0)] = 0, \qquad l_4^{h\tau}[r_i v_{iN} - r(x_i)v(x_2,1)] = 0.$$

根据引理 8 得：当$\tau\leqslant C_1$时，$|r(x_i)[v(x_i,y_j) - v_{ij}]| \leqslant C(h+\tau)$；当$\tau\geqslant C_1$时，由式(13) 推出$|L^{h\tau}[r(x_i)v_{ij}]| \leqslant C$。又

$$l_1^{h\tau}[r_0 v_{0j}] = l_1[r(0)v(0,y_j)] = ar(0)v(0,y_j) - \beta\frac{\partial[r(x)v(x,y_j)]}{\partial x}$$

所以

$$|l_1^{h\tau}[r_0 v_{0j}]| \leqslant C.$$

同理

$$|l_1^{h\tau}[r_N v_{Nj}]| \leqslant C, \qquad |l_3^{h\tau}[r_i v_{i0}]| \leqslant C, \qquad |l_4^{h\tau}[r_i v_{iM}]| \leqslant C。$$

根据引理 8 知$|r(x_1)v_{ij}| \leqslant C$，故当$\tau\geqslant C_1$时，

$$|r(x_i)[v(x_i,y_j) - v_{ij}]| \leqslant C(h+\tau).$$

引理10　若z_{ij}是方程(12)的解，则$|z(x_i,y_j) - z_{ij}| \leqslant C(h+\tau)$。

证明

$$L^{h\tau}[z_{ij} - z(x_i,y_j)] = Lz(x_i,y_j) - L^{h\tau}z(x_i, y_j)$$

$$= \varepsilon[\frac{\partial^2 z(x_i,y_j)}{\partial y^2} - \delta_y^2 z(x_i,y_j)] + \varepsilon(1-\sigma_{ij})\delta_y^2 z(x_i,y_j)$$

$$- a(x_i,y_j)\cdot[\frac{\partial z(x_i,y_j)}{\partial y} - D_{0y}z(x_i,y_j)] + \frac{\partial^2 z(x_i,y_j)}{\partial x^2} - \delta_x^2 z(x_i,y_j),$$

$$|L^{h\tau}[z(x_i,y_j) - z_{ij}]|$$

$$\leqslant C\cdot\int_{y_{j-1}}^{y_{j+1}}[\varepsilon\left|\frac{\partial^3 z(x_i,s)}{\partial y^3}\right| + \left|\frac{\partial^2 z(x_i,s)}{\partial y^2}\right|]ds + Ch^2$$

$$\leqslant C\tau + C\exp(-\frac{a(1-y_j)}{\varepsilon})st(a\rho) + Ch^2$$

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$$\leqslant C\tau\left\{1+\frac{1}{max(\tau,\varepsilon)}exp\left(-\frac{a(1-y_{j+1})}{\varepsilon}\right)\right\}+Ch^2,$$

$$\left|l_1^{h\tau}[z(0,y_j)-z_{0j}]\right|=\beta\left|\frac{\partial z(0,y_j)}{\partial x}-\frac{z(h,y_j)-z(0,y_j)}{h}\right|\leqslant Ch.$$

同理

$$\left|l_2^{h\tau}[z(1,y_j)-z_{Nj}]\right|\leqslant Ch,$$

$$l_3^{h\tau}[z(x_i,0)-z_{i0}]=0,\qquad l_4^{h\tau}[z(x_i1)-z_{iM}]=0.$$

由引理 8 知$|z(x_i,y_j)-z_{ij}|\leqslant C(h+\tau)$.

综合引理 9 和引理10，易得

定理 若$u(x_i,y_j)$、u_{ij}分别是微分问题(1)－(5)和差分格式(6)—(10)的解，则

$$|u(x_i,y_j)-u_{ij}|\leqslant C(h+\tau).$$

3 数值例子

在正方形区域｛$0\leqslant x,\ y\leqslant 1$｝考虑方程

$$\varepsilon\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial y}=2(y^4-2y^3)-(4y^3-6y^2)(x^2-x),$$

$$\left[u-\frac{\partial u}{\partial x}\right]_{x=0}=y^4-2y^3,\qquad \left[u+\frac{\partial u}{\partial x}\right]_{x=1}=y^4-2y^3,$$

$$u|_{y=0},\qquad u|_{y=1}=x^4-2x^3-7x^2+8x+7,$$

容易证明该方程满足相容性条件．直接解退化方程得

$$w(x,y)=(y^4-2y^3)(x^2-x),$$

根据引理 3，求出问题的渐近解为

$$u(x,y)=w(x,y)+(x^4-2x^3-6x^2+7x+7)\exp\left(-\frac{1-y}{\varepsilon}\right)+0(\varepsilon).$$

在计算机上解差分方程组，将计算结果列于表 1，与渐近解相比较，易见和理论相符合．

表1 $H=1/32$时渐近解、数值解和误差表

$h=1/32$	点坐标	渐近解	数值解	误 差*
	(32/32, 32/32)	7	7	0
	(32/32, 31/32)	0.3075586	0.3039912	3.567333E-3
	(32/32, 30/32)	1.351318E-2	1.190392E-2	1.609263E-3
	(32/32, 5/32)	0	−7.416785E-5	7.416785E-5
$\varepsilon=10^{-2}$	(31/32, 31/32)	0.3452928	0.3427693	2.503157E-3
	(16/32, 31/32)	0.6215843	0.6197856	1.798689E-3
	(16/32, 17/32)	5.505347E-2	5.509005E-2	−3.657863E-5
	(16/32, 1/32)	1.502037E-5	1.794546E-5	−2.925091E-6
	(0, 17/32)	0	−3.811893E-4	3.811893E-4

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续表1

$h=1/32$	点坐标	渐近解	数值解	误　差*
	(0 , 1)	7	7	0
	(0 , 31/32)	0	$-8.878419E-4$	$8.878419E-4$
	(0 , 30/32)	0	$-8.270464E-4$	$8.270464E-4$
	(0 , 5/32)	0	$-6.660368E-6$	$6.660368E-6$
$\varepsilon=10^{-6}$	(1/32, 31/32)	$2.838317E-2$	$2.793982E-2$	$4.433487E-4$
	(16/32, 31/32)	0.23439	0.2343918	$-1.817942E-6$
	(16/32, 17/32)	$5.505347E-2$	$5.506476E-2$	$-1.12839E-5$
	(16/32, 1/32)	$1.502037E-5$	$1.591342E-5$	$-8.93047E-7$
	(32/32, 5/32)	0	$-5.41859E-6$	$5.41859E-6$

*表中误差是指渐近解减去数值解.

表 2　$\varepsilon=10^{-4}$时渐近解数值解和误差表

$\varepsilon=10^{-4}$	点坐标	渐近解	数值解	误　差*
	(0 , 1)	7	7	0
	(0 , 15/16)	0	$-6.658112E-3$	$6.658112E-3$
	(0 , 14/16)	0	$-5.717257E-3$	$5.717257E-3$
	(0 , 5/16)	0	$-3.486922E-4$	$3.486922E-4$
$h=1/16$	(2/16, 5/16)	$9.575486E-2$	$9.289792E-2$	$2.85694E-3$
	(8/16, 15/16)	0.2183683	0.2188241	$4.413724E-5$
	(8/16, 7/16)	0.0128746	0.0131368	$-2.621934E-4$
	(8/16, 1/16)	$1.182556E-4$	$1.645084E-4$	$-4.625278E-5$
	(16/16, 5/16)	0	$-2.577018E-4$	$2.577018E-4$
	(0 , 1)	7	7	0
	(0 , 63/64)	0	$-2.3228744E-4$	$2.328744E-4$
	(0 , 62/64)	0	$-2.253752E-4$	$2.253752E-4$
	(0 , 20/64)	0	$-1.237943E-5$	$1.237943E-5$
$h=1/64$	(8/64, 1/64)	$1.490032E-2$	$1.478392E-2$	$1.164051E-4$
	(32/64, 63/64)	0.2421894	0.2421895	$-1.043081E-7$
	(32/64, 20/64)	0.0128746	$1.287582E-2$	$-1.214446E-6$
	(32/64, 4/64)	$1.182556E-4$	$1.185624E-4$	$-3.068126E-7$
	(1, 20/64)	0	$-1.22366E-5$	$1.22366E-5$

*表中误差是指渐近解减去数值解.

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Numerical Solution to an Elliptic-Parabolic Partial Differential Equation with Singular Perturbation and Mixed Boundary Condition

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Abstract This paper gives a numerical solution to an elliptic-parabolic differential equation with singular pertuibation and mixed boundarp condition. It constructs a difference scheme of first order and uniform convergence.

Key words elliptic-parabolic partial differential, singular pertubation, mixed boundary condition, uniform convergence

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