

# 以应力函数导出细长开口薄壁 圆管的扭转应力

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## 摘 要

承受载荷的壳体,由于薄膜比性的作用而引起其分析的复杂性,本文以应力函数简捷地导出细长开口薄壁圆管的扭转剪应力,且其近似的应力、内力公式与初步近似的薄壳理论结果一致。

## 一、细长开口薄壁圆管的扭转剪应力

### 1. 扭转剪应力 $\tau_{zx}$

在自由扭转的条件下,由弹性力学等截面直杆的扭转理论得

$$\nabla^2 \varphi = C, \quad (1)$$

$$\varphi_s = 0, \quad (2)$$

$$\tau_{zx} = \frac{\partial \varphi}{\partial y} \quad (3)$$

$$\tau_{zy} = -\frac{\partial \varphi}{\partial x}$$

从薄膜比拟的分析指出,在横截面距开口端弧长(壁厚中心线)大于 $2t$ 的区域,应力分布与极坐标的极角 $\theta$ 无关,只与极径 $r$ 有关,现以极坐标表示式(1)、(2)、(3)得

$$\nabla^2 \varphi = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \varphi = C,$$

因应力分布与极角 $\theta$ 无关,故

$$\nabla^2 \varphi = \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \varphi = C, \quad (4)$$

$$[\varphi]_{r=r_1} = [\varphi]_{r=r_2} = 0, \quad (5)$$

如图1所示可得

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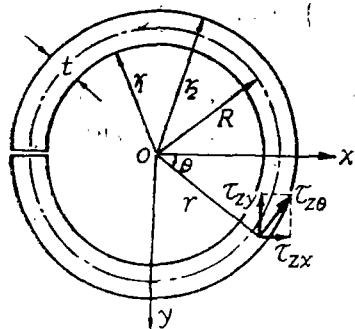


图 1

$$\tau_{z\theta} = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \sqrt{\left(\frac{\partial \varphi}{\partial y}\right)^2 + \left(\frac{\partial \varphi}{\partial x}\right)^2}, \quad (6a)$$

$$\frac{\partial \varphi}{\partial y} = \sin \theta \frac{\partial \varphi}{\partial r} + \frac{\cos \theta}{r} \frac{\partial \varphi}{\partial \theta} = \sin \theta \frac{d\varphi}{dr} \quad (6b)$$

$$\frac{\partial \varphi}{\partial x} = \cos \theta \frac{\partial \varphi}{\partial r} - \frac{\sin \theta}{r} \frac{\partial \varphi}{\partial \theta} = \cos \theta \frac{d\varphi}{dr}$$

将式(6b)代入式(6a)得

$$\tau_{z\theta} = \frac{d\varphi}{dr}. \quad (7)$$

取应力函数为

$$\varphi = \frac{C}{4} \left[ (r^2 - r_1^2) + \frac{r_2^2 - r_1^2}{\ln\left(\frac{r_2}{r_1}\right)} (\ln r_1 - \ln r) \right], \quad (8)$$

式中,  $\varphi$  为应力函数;  $C$  为常数;  $r$  为极径(图 1);  $r_1$  为薄壁管内半径(图 1);  $r_2$  为薄壁管外半径(图 1);  $R$  为管壁厚中心线半径(图 1);  $t$  为管壁厚度(图 1)。

将式(8)代入式(4)和(5), 知能满足其等式的要求, 故式(8)为薄壁管扭转时的应力函数。

由式(7)得剪应力

$$\tau_{z\theta} = \frac{d\varphi}{dr} = \frac{C}{4} \left[ 2r - \frac{r_2^2 - r_1^2}{r \ln\left(\frac{r_2}{r_1}\right)} \right]. \quad (9)$$

## 2. $\tau_{z\theta}$ 的近似公式

如图 1 所示,  $r_1 = R - t/2$ ,  $r_2 = R + t/2$ ,  $r = R + \zeta$ , 代入式(9)得

$$\begin{aligned} \tau_{z\theta} &= \frac{C}{4} \left[ 2(R + \zeta) - \frac{(R + \frac{t}{2})^2 - (R - \frac{t}{2})^2}{(R + \zeta) \ln\left(\frac{R + \frac{t}{2}}{R - \frac{t}{2}}\right)} \right] \\ &= \frac{CR}{2} \left[ 1 + \frac{\zeta}{R} - \frac{\frac{t}{R}}{(1 + \frac{\zeta}{R}) \ln\left(\frac{1 + \frac{t}{2R}}{1 - \frac{t}{2R}}\right)} \right] \\ &= \frac{CR}{2} \left\{ 1 + \frac{\zeta}{R} - \frac{\frac{t}{R}}{(1 + \frac{\zeta}{R})^2 \left[ -\frac{t}{2R} + \frac{1}{3} \left(\frac{t}{2R}\right)^3 + \frac{1}{5} \left(\frac{t}{2R}\right)^5 + \dots \right]} \right\} \\ &= \frac{CR}{2} \left[ 1 + \frac{\zeta}{R} - \frac{1}{1 + \frac{\zeta}{R} + \frac{1}{3} \left(\frac{t}{2R}\right)^2 \left(1 + \frac{\zeta}{R}\right) + \frac{1}{5} \left(\frac{t}{2R}\right)^4 \left(1 + \frac{\zeta}{R}\right) + \dots} \right] \end{aligned}$$

$$\approx \frac{CR}{2} \left[ \frac{(1 + \frac{\zeta}{R})^2 - 1}{1 + \frac{\zeta}{R}} \right] = \frac{CR}{2} \times 2 \frac{\zeta}{R} \left( \frac{1 + \frac{\zeta}{R}}{1 + \frac{\zeta}{R}} \right) \approx C\zeta, \quad (10)$$

式中,  $\zeta$  为由中心线起计量的径向坐标。因  $(t/2R) \ll 1$ ,  $(\zeta/R) \ll 1$ , 在计量过程中可略去高阶微量。式(10)指出, 剪应力沿壁厚近似于线性变化(图2)。

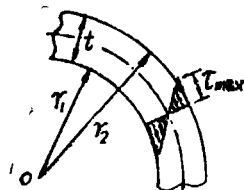


图 2

### 3. 常数C的确定

常数C与外加的扭矩T有关, 由剪应力 $\tau_{z\theta}$ 对圆管中心取矩得

$$T = 2\pi \int_{r_1}^{r_2} r^2 \tau_{z\theta} dr \approx 2\pi \int_{-t/2}^{t/2} (R + \zeta)^2 C \zeta d\zeta = \frac{\pi CR t^3}{3},$$

故

$$T \approx \frac{\pi CR t^3}{3}, \quad (11a)$$

$$C \approx \frac{3T}{\pi R t^3}. \quad (11b)$$

### 4. $\tau_{z\theta}$ 的最大值

$$|\tau_{z\theta}|_{\max} = [\tau_{z\theta}]_{r=r_2} \approx [C\zeta]_{\zeta=t/2} = C \frac{t}{2} = \frac{3}{2} \frac{T}{\pi R t^2},$$

故

$$|\tau_{z\theta}|_{\max} \approx \frac{3}{2} \frac{T}{\pi R t^2}. \quad (12)$$

## 二、沿管壁厚度剪应力的合力

将剪应力 $\tau_{z\theta}$ 向半径为R的中心线简化成为一个单位弧长上的扭矩 $M_{z\theta}$ 和单位弧长上的剪力 $N_{z\theta}$ 。

$$\begin{aligned} M_{z\theta} &= \int_{r_1}^{r_2} \frac{(r-R)\tau_{z\theta} r d\theta dr}{R d\theta} = \frac{1}{R} \int_{r_1}^{r_2} r(r-R)\tau_{z\theta} dr \\ &\approx \frac{1}{R} \int_{-t/2}^{t/2} (R+\zeta)(R+\zeta-R)C\zeta d\zeta = \frac{T}{4\pi R}, \end{aligned}$$

即

$$M_{z\theta} \approx \frac{T}{4\pi R}. \quad (13)$$

同理

$$N_{z\theta} = \int_{r_1}^{r_2} \frac{\tau_{z\theta} r d\theta dr}{R d\theta} = \frac{1}{R} \int_{r_1}^{r_2} r \tau_{z\theta} dr \approx \frac{1}{R} \int_{-t/2}^{t/2} (R+\zeta)C\zeta d\zeta = \frac{T}{4\pi R^2},$$

即

$$N_{z\theta} \approx \frac{T}{4\pi R^2}. \quad (14)$$

式(13)和(14)指出, 细长开口薄壁圆管的外扭矩T正比于截面壁厚中心线单位弧长上的内力扭矩 $M_{z\theta}$ 和剪力 $N_{z\theta}$ 。

### 三、与初步近似薄壳理论比较

在相同条件下,按初步近似薄壳理论得

$$M_{z\theta} = \frac{T}{4\pi R}, \quad (15)$$

$$N_{z\theta} = \frac{T}{4\pi R^2}, \quad (16)$$

式(13)和(14)各与初步近似薄壳理论的式(15)和(16)吻合。

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## Driving The Torsional Stress of a Slender Slit Thin-Walled Round Tube with Stress Function

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### Abstract

The loaded shell shows its complexity in analysis due to the effect of its thin wall and flexibility. By means of stress function, the torsional stress of a slender slit thin-walled round tube was derived in this paper simply and rapidly, and its approximate formulas of stress and internal forces coincide with the result of the preliminary approximate shell theory.