弹 性 板 补 充 理 论

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为了补充A. A. Kromm. 的厚板理论（$\varepsilon_{zz}=0$），本文作以下分析：

（1）几何方程：

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

（2）物理方程：

$$\sigma_{ij}=\lambda\theta\delta_{ij}+2\mu e_{ij} \tag{2}$$

（3）平衡方程：

$$\frac{\partial\sigma_{ij}}{\partial x_j}+F_i=0 \tag{3}$$

（4）边界条件：

$$R_i=\sigma_{ij}l_{nj} \tag{4}$$

现在用直角坐标系，原点放在均匀厚板的形心上，设厚板厚度方向为z轴，若以右手坐标系，则x、y轴在厚板中性层平面上，如果除厚板支座支承边界条件外，在厚板$z=-\frac{h}{2}$板面上，有与z轴方向一致的荷重$q(x,y)$。在厚板$z=\frac{h}{2}$板面上及四周无其它荷重情况下，则边界条件（4）为

$$R=q(x,y)$$

现令

$$\sigma_{zz}=q(x,y)f(z) \tag{5}$$

可以根据边界条件（4）

$$0=q(x,y)f\left(\frac{h}{2}\right) \qquad \therefore\ f\left(\frac{h}{2}\right)=0 \tag{6}$$

$$-q(x,y)=q(x,y)f\left(-\frac{h}{2}\right) \qquad \therefore\ f\left(-\frac{h}{2}\right)=-1 \tag{7}$$

又令

$$\begin{aligned}\sigma_{zy}&=Q_x\varphi(z),\\ \sigma_{zy}&=Q_y\varphi(z)\end{aligned} \tag{8}$$

根据边界条件σ_{xz}及σ_{yz}在厚板上下面皆为零

$$\therefore\ \varphi\left(\pm\frac{h}{2}\right)=0 \tag{9}$$

从Q_x，Q_y与σ_{xz}，σ_{yz}自身平衡关系，

由（8）得

$$Q_x=\int_{-\frac{1}{2}h}^{+\frac{1}{2}h}\sigma_{xz}dz=\int_{-\frac{1}{2}h}^{+\frac{1}{2}h}Q_x\varphi(z)dz,$$

$$Q_y=\int_{-\frac{1}{2}h}^{+\frac{1}{2}h}\sigma_{yz}dz=\int_{-\frac{1}{2}h}^{+\frac{1}{2}h}Q_y\varphi(z)dz,$$

可得

$$\int_{-\frac{1}{2}h}^{+\frac{1}{2}h}\varphi(z)dz=1 \tag{10}$$

由（5）、（8）代入（3）

得

$$\varphi(z)\frac{\partial Q_x}{\partial x}+\varphi(z)\frac{\partial Q_y}{\partial y}+q(x,y)f'(z)=0 \tag{11}$$

厚板平衡方程：

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q(x,y)=0 \tag{12}$$

如果

$$f'(z)=\varphi(z) \tag{13}$$

$$\int_{-\frac{1}{2}h}^{+\frac{1}{2}h}f'(z)dz=\left|f(z)\right|_{-\frac{1}{2}h}^{+\frac{1}{2}h}=f\left(\frac{h}{2}\right)-f\left(-\frac{h}{2}\right)=0-(-1)=0$$

满足(10)，则(12)成立，

因此

$$\sigma_{xz}=Q_xf'(z)\qquad \sigma_{yz}=Q_yf'(z) \tag{14}$$

如果，令

$$\omega=\omega_0\psi(z) \tag{15}$$

式中 ω_0 为厚板中层面的挠度

从 $\psi(z)$ 函数可以断定，$\psi(0)=1$，另外 $\psi(z)$ 自身必是偶函数，现在如何求 $\psi(z)$，$\varphi(z)$ 及 $f(z)$ 的关系式

$$\psi(0)=1 \tag{16}$$

$$\varepsilon_{zz}=\frac{\partial\omega}{\partial z}=\omega_0\psi'(z) \tag{17}$$

由（1）

$$\frac{\partial u}{\partial z}+\frac{\partial\omega}{\partial x}=\frac{\sigma_{xz}}{\mu}$$

$$\frac{\partial v}{\partial z}+\frac{\partial\omega}{\partial y}=\frac{\sigma_{yz}}{\mu}$$

$$\left.\begin{aligned}\frac{\partial u}{\partial z}&=-\frac{\partial\omega}{\partial x}+\frac{1}{\mu}Q_xf'(z)=-\frac{\partial\omega_0}{\partial x}\psi(z)+\frac{Q_x}{\mu}f'(z)\\ \frac{\partial v}{\partial z}&=-\frac{\partial\omega}{\partial y}+\frac{1}{\mu}Q_yf'(z)=-\frac{\partial\omega_0}{\partial y}\psi(z)+\frac{Q_y}{\mu}f'(z)\end{aligned}\right\} \tag{18}$$

积分

$$\left.\begin{aligned}u&=-\frac{\partial\omega_0}{\partial x}\int\psi(z)dz+\frac{1}{\mu}Q_xf(z)+u_0\\ v&=-\frac{\partial\omega_0}{\partial y}\int\psi(z)dz+\frac{1}{\mu}Q_yf(z)+v_0\end{aligned}\right\} \tag{19}$$

中层面上除 z 方向有位移外，x 及 y 方向无位移，

$$\therefore \quad y=0 \text{ 时} \qquad u=v=0$$

代入（19）

$$-\frac{\partial\omega_0}{\partial x}\int\psi(0)dz+\frac{Q_x}{\mu}f(0)+u_0=0$$

$$-\frac{\partial\omega_0}{\partial y}\int\psi(0)dz+\frac{Q_y}{\mu}f(0)+v_0=0$$

$$\left.\begin{aligned}u_0&=\frac{\partial\omega_0}{\partial x}z-\frac{Q_x}{\mu}f(0)\\ v_0&=\frac{\partial\omega_0}{\partial y}z-\frac{Q_y}{\mu}f(0)\end{aligned}\right. \tag{20}$$

将（20）代入（19）得

$$\left.\begin{aligned}u&=-\frac{\partial\omega_0}{\partial x}\left[\int\psi(z)\,dz-z\right]+\frac{Q_x}{\mu}\left[f(z)-f(0)\right]\\ v&=-\frac{\partial\omega_0}{\partial y}\left[\int\psi(z)\,dz-z\right]+\frac{Q_y}{\mu}\left[f(y)-f(0)\right]\end{aligned}\right. \tag{21}$$

将（21）代入（1）

$$\left.\begin{aligned}\varepsilon_{xx}&=\frac{\partial u}{\partial x}=-\frac{\partial^2\omega_0}{\partial x^2}\left[\int\psi(z)\,dz-z\right]+\frac{1}{\mu}\frac{\partial Q_x}{\partial x}\left[f(z)-f(0)\right]\\ \varepsilon_{yy}&=\frac{\partial v}{\partial y}=-\frac{\partial^2\omega_0}{\partial y^2}\left[\int\psi(z)\,dz-z\right]+\frac{1}{\mu}\frac{\partial Q_y}{\partial y}\left[f(z)-f(0)\right]\\ \varepsilon_{zz}&=\frac{\partial\omega}{\partial z}=\omega_0\,\psi'(z)\end{aligned}\right\} \tag{22}$$

由（22）、（11）及（12）

$$\begin{aligned}\theta&=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}\\ &=-\nabla^2\omega_0\left[\int\psi(z)dz-z\right]+\frac{1}{\mu}\left(\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}\right)\left[f(z)-f(0)\right]+\omega_0\psi'(z)\\ &=-\nabla^2\omega_0\left[\int\psi(z)dz-z\right]-\frac{1}{\mu}q(x,y)\left[f(z)-f(0)\right]+\omega_0\psi'(z)\end{aligned} \tag{23}$$

将（23）代入（2）

$$\begin{aligned}q(x,y)f(z)=&-\lambda\nabla^2\omega_0\left[\int\psi(z)\,dz-z\right]-\frac{\lambda}{\mu}q(x,y)\left[f(z)-f(0)\right]+\lambda\omega^0\psi'(z)\\ &+2\mu\omega^0\psi'(z)\end{aligned}$$

得

$$q(x,y)f(z)\left[1+\frac{\lambda}{\mu}\right]=-\lambda\nabla^2\omega_0\left[\int\psi(z)\,dz-z\right]+\frac{\lambda}{\mu}q(x,y)f(0)+(\lambda+2\mu)\omega^0\psi'(z)$$

等式两边对子求导，

$$(1+\frac{\lambda}{\mu})q(x,y)\,f'(z)=-\lambda\nabla^2\omega^0\left[\psi(z)-1\right]+\frac{\lambda}{\mu}q(x,y)\,f'(0)+(\lambda+2\mu)\omega_0\psi''(z)$$

化简得

$$\psi''(z)-\frac{\lambda\nabla^2\omega_0}{(\lambda+2\mu)\omega_0}\psi(z)=\frac{(1+\frac{\lambda}{\mu}q(x,y)f'(z)-\lambda\nabla^2\omega_0-\frac{\lambda}{\mu}q(x,y)f'(0)}{(\lambda+2\mu)\omega_0}$$

令 $k^2=\frac{\lambda\nabla^2\omega_0}{(\lambda+2\mu)\omega_0}$, $A=\frac{(1+\frac{\lambda}{\mu})q(x_1y)}{(\lambda+2\mu)\omega_0}$, $C=\frac{\frac{\lambda}{\mu}q(x_1y)f'(0)}{(\lambda+2\mu)\omega_0}$

则得

$$\psi''(z)-k^2\psi(z)=A\varphi(z)-k^2-C \tag{24}$$

其解为

$$\psi(z)=e^{-kz}\int\left\{\int\left[A\phi(z)-k^2-C\right]e^{-kz}dz\right\}e^{2kz}dz+Ge^{kz}+He^{-kz} \tag{25}$$

$$f(z)=\int\varphi(z)dz+L(x_1y) \tag{26}$$

式中函数 $\varphi(z)$ 在 $-\frac{h}{2}\leqslant z\leqslant+\frac{h}{2}$ 上连续可积，

由（2）求应力分量

$$\sigma_{xx}=\lambda\theta+2\mu\varepsilon_{xx}=-\left[\lambda\Delta^2\omega_0+2\mu\frac{\partial^2\omega_0}{\partial x^2}\right]\left[\int\psi(z)dz-z\right]$$
$$-\left[\frac{\lambda}{\mu}q(x,y)-2\frac{\partial Q_x}{\partial x}\right]\left[f(z)-f(0)\right]+\lambda\omega_0\psi'(z) \tag{27}$$

$$\sigma_{yy}=\lambda\theta+2\mu\varepsilon_{yy}=-\left[\lambda\nabla^2\omega_0+2\mu\frac{\partial^2\omega_0}{\partial y^2}\right]\left[\int\psi(z)dz-z\right]$$
$$-\left[\frac{\lambda}{\mu}q(x,y)-2\frac{\partial Q_y}{\partial y}\right]\left[f(z)-f(0)\right]+\lambda\omega_0\psi'(z) \tag{28}$$

$$\sigma_{zz}=q(x,y)f(z) \tag{29}$$

$$\sigma_{xy}=\mu\varepsilon_{xy}=-2\mu\frac{\partial^2\omega_0}{\partial x\partial y}\left[\int\psi(z)dz-z\right]+\left(\frac{\partial Q_x}{\partial y}+\frac{\partial Q_y}{\partial x}\right)\left[f(z)-f(0)\right] \tag{30}$$

$$\sigma_{yz}=Q_yf'(z) \tag{31}$$

$$\sigma_{xz}=Q_xf'(z) \tag{32}$$

将（27）、（30）、（32）代入（3），不计厚板体力

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}=0$$

得

$$-\left[\lambda\frac{\partial}{\partial x}(\nabla^2\omega_0+2\mu\frac{\partial^3\omega_0}{\partial x^3}\right]\left[\int\psi(z)dz-z\right]-\left[\frac{\lambda}{\mu}\frac{\partial q(x,y)}{\partial x}-2\frac{\partial^2Q_x}{\partial x^2}\right]\left[f(z)-f(0)\right]$$
$$+\lambda\frac{\partial\omega_0}{\partial x}\psi'(z)-2\mu\frac{\partial^3\omega_0}{\partial x\partial y^2}\left[\int\psi(z)dz-z\right]+\left(\frac{\partial^2Q_x}{\partial q^2}+\frac{\partial^2Q_y}{\partial x\omega y}\right)\left[f(z)-f(0)\right]$$
$$+Q_xf''(z)=0 \tag{33}$$

将（28），（30），（31）代入

$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}=0$$

得

$$-2\mu\frac{\partial^3\omega_0}{\partial x^2\partial y}\left[\int\psi(z)dz-z\right]+\left(\frac{\partial^2Q_x}{\partial x\partial y}\right)\left[f(z)-f(0)\right]$$

$$-\left[\lambda\frac{\partial}{\partial y}(\nabla^2\omega_0)+2\mu\frac{\partial^3\omega_0}{\partial y^3}\right]\left[\int\psi(z)dz-z\right]-\left[\frac{\lambda}{\mu}\frac{\partial q(x,y)}{\partial y}-2\frac{\partial^2Q_y}{\partial y^2}\right]\left[f(z)-f(0)\right]$$

$$+\lambda\frac{\partial\omega_0}{\partial y}\psi'(z)+Q_Xf''(z)=0 \tag{34}$$

由
$$\frac{\partial}{\partial x}\left[(33)\right]+\frac{\partial}{\partial y}\left[(34)\right]$$

得
$$-(\lambda+2\mu)\nabla^2\nabla^2\omega_0\left[\int\psi(z)dz-z\right]+\lambda\nabla^2\omega_0\psi'(z)$$

$$-\frac{\lambda+2\mu}{\mu}\nabla^2q(x_1y)\left[f(z)-f(0)\right]-q(x_1y)f''(z)=0 \tag{35}$$

如果不考虑 $\psi(z)$，则得

$$(\lambda+2\mu)z\nabla^2\nabla^2\omega_0-\frac{\lambda+2\mu}{\mu}\nabla^2q(x,y)\left[f(z)-f(0)\right]-q(x,y)f''(z)=0 \tag{36}$$

在（36）中常数 $1-2\nu=(1-\nu)^2$ 时即为A. A. Kromm. 公式，

公式（35）中 $\psi(z)$ 与 $f(z)$，可以（25），（26）式中函数决定。$f(z)$ 满足

$$f\left(-\frac{h}{2}\right)=-1 \qquad f\left(\frac{h}{2}\right)=0 \qquad f'\left(\pm\frac{h}{2}\right)=0$$

在 $\psi(z)$ 中的 $\psi(0)=1$，$\psi(z)$ 为偶函数。

由（33）、（34）、（35）可以求出未知函数 Q_x，Q_y及ω_0，而$f(z)$可根据荷重及边界条件先确定之，三个方程，三个未知数是可以解的。

举例

设 $q(x,y)=q$，是厚板 $z=-\frac{h}{2}$的板面上，与z轴方向一致的均布荷重 q。

令
$$f(z)=-\frac{1}{4}\left[2-3\left(\frac{2z}{h}\right)+\left(\frac{2z}{h}\right)^3\right]$$

它满足

$$\sigma_{zz}=qf(z) \qquad f\left(\frac{h}{2}\right)=0 \qquad f\left(-\frac{h}{2}\right)=-1$$

$$f'(z)=\frac{3}{2h}\left[1-\left(\frac{2z}{h}\right)^2\right]=\varphi(z), \qquad \text{它满足：} f'\left(\pm\frac{h}{2}\right)=0$$

$$f''(z)=-\frac{12}{h^3}z$$

$$f(z)-f(0)=\frac{z}{2h}\left[3-\left(\frac{2z}{h}\right)^2\right]$$

$$\psi(z)=e^{-kz}\int\left[\int(A\varphi(z)-k^2-C)e^{-kz}dz\right]e^{2kz}dz$$

$$=1-\frac{3A}{2hk^2}-\frac{12A}{h^3k^4}+\frac{C}{k^2}+\frac{6A}{h^3k^2}z^2 \text{ 为偶函数，}$$

由于函数 $\psi(z)$ 必须满足 $\psi(0)=1$ 故得

$$\frac{-3A}{2hk^2}+\frac{C}{k^2}-\frac{12A}{h^3k^4}=0 \tag{37}$$

因为 $A=\dfrac{(1+\frac{\lambda}{\mu}q}{(\lambda+2\mu)\omega_0}$　　$k^2=\dfrac{\lambda\nabla^2\omega_0}{(\lambda+2\mu)\omega_0}$　　$C=\dfrac{\lambda qf'(0)}{\mu(\lambda+2\mu)\omega_0}$

代入上式（37），简化得

$$k^2\lambda\mu\nabla^2\omega_0+8(\lambda+\mu)(\lambda+2\mu)\omega_0=0$$

即

$$\nabla^2\omega_0+\frac{8(\lambda+\mu)(\lambda+2\mu)}{h^2\lambda\mu}\omega_0=0$$

令

$$n^2=\frac{8(\lambda+\mu)(\lambda+2\mu)}{h^2\lambda\mu}$$

$$\therefore \quad \nabla^2\omega_0+n^2\omega_0 0 \tag{38}$$

若为矩形板可令

$$\omega_0=H\cos(nx+\delta)+H\cos(ny+\delta) \tag{39}$$

可以满足（38），式中 H 及 δ 为积分常数，

若为圆形板可令 ω_0 极坐标形式解

将 A，k^2，C 代入 $\psi(z)$ 式，得

$$\psi(z)=1+\frac{3hq}{16(\lambda+2\mu)\omega_0}-\frac{3h\lambda q}{16(\lambda+\mu)(\lambda+2\mu)\omega_0}-\frac{12h\mu q}{64(\lambda+\mu)(\lambda+2\mu)\omega_0}+\frac{6qz^2}{16(\lambda+2\mu)\omega_0}$$

令

$$M=\frac{3hq}{16(\lambda+2\mu)\omega_0}-\frac{3h\lambda q}{16(\lambda+\mu)(\lambda+2\mu)\omega_0}-\frac{12h\mu q}{64(\lambda+\mu)(\lambda+2\mu)\omega_0}$$

所以

$$\psi(z)=1+M+\frac{3q}{4h(\lambda+2\mu)\omega_0}z^2$$

$$\psi'(z)=\frac{3q}{2h(\lambda+2\mu)\omega_0}z$$

$$\int\psi(z)dz-z=Mz+\frac{q}{4h(\lambda+2\mu)\omega_0}z^3$$

由（38）代入（35）可以满足

故

$$\omega_0=H\cos(nx+\delta)+H\cos(ny+\delta)$$

为（35）解，　H 及 δ 为积分常数

$$\frac{\partial\omega_0}{\partial x}=-Hn\sin(nx+\delta) \qquad \frac{\partial^2\omega_0}{\partial x^2}=-Hn^2\cos(nx+\delta)$$

$$\frac{\partial\omega_0}{\partial y}=-Hn\sin(ny+\delta) \qquad \frac{\partial^2\omega_0}{\partial y^2}=-Hn^2\mathrm{con}(ny+\delta)$$

$$\nabla^2\omega_0=-Hn^2\Big[\cos(nx+\delta)+\cos(ny+\delta)\Big]$$

$$\frac{\partial}{\partial x}(\nabla^2\omega_0)=Hn^3\sin(nx+\delta) \qquad \frac{\partial}{\partial y}(\nabla^2\omega_0)=Hn^3\sin(ny+\delta)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \omega_0}{\partial x^2}\right)=Hn^3\sin(nx+\delta)\qquad \frac{\partial}{\partial y}\left(\frac{\partial^2 \omega_0}{\partial y^2}\right)=Hn^3\sin(ny+\delta)$$

从（12）得

$$\frac{\partial^2 Q_x}{\partial x^2}+\frac{\partial^2 Q_y}{\partial x\partial y}=0\quad 及\quad \frac{\partial^2 Q_x}{\partial x\partial y}+\frac{\partial^2 Q_y}{\partial y^2}=0$$

代入（33）及34）得

$$-\nabla^2 Q_x\left[f(z)-f(0)\right]+Q_x f''(z)=\left[\lambda\frac{\partial}{\partial x}\nabla^2\omega_0+2\mu\frac{\partial^3\omega_0}{\partial x^3}\right]\left[\int\psi(z)\,dz-z\right]-$$

$$-\lambda\frac{\partial\omega_0}{\partial x}\psi'(z)+2\mu\frac{\partial^3\omega_0}{\partial x\partial y^2}\left[\int\psi(z)dz-z\right]$$

$$-\nabla^2 Q_y\left[f(z)-f(0)\right]+Q_y f''(z)=\left[\lambda\frac{\partial}{\partial y}\nabla^2\omega_0+2\mu\frac{\partial^3\omega_0}{\partial y^3}\right]\left[\int\psi(z)dz-z\right]-$$

$$-\lambda\frac{\partial\omega_0}{\partial y}\psi'(z)+2\mu\frac{\partial^3\omega_0}{\partial y\partial x^2}\left[\int\psi(z)dz-z\right]$$

代入后再整理得

$$\nabla^2 Q_x+\frac{6}{3h^2-4z^2}Q_x=$$

$$=\frac{\sin(ny+\delta)}{\cos(nx+\delta)+\cos(ny+\delta)}\left\{\frac{n^3(\lambda+2\mu)\left[M'+\dfrac{qz^2}{4h(\lambda+2\mu)}\right]+\dfrac{3q\lambda n}{2h(\lambda+2\mu)}}{-\left(\dfrac{3}{2h}-\dfrac{2z^2}{h^2}\right)}\right\}$$

$$\nabla^2 Q_y+\frac{6}{3h^2-4z^2}Q_y=$$

$$=\frac{\sin(ny+\delta)}{\cos(nx+\delta)+\cos(ny+\delta)}\left\{\frac{n^3(\lambda+2\mu)\left[M'+\dfrac{qz^2}{4h(\lambda+2\mu)}\right]+\dfrac{3q\lambda n}{2h(\lambda+2\mu)}}{-\left(\dfrac{3}{2h}+\dfrac{2z^2}{h^2}\right)}\right\}$$

式中
$$M'=\frac{3hq}{16(\lambda+2\mu)}-\frac{3h\lambda q}{16(\lambda+\mu)(\lambda+2\mu)}-\frac{12h\mu q}{64(\lambda+\mu)(\lambda+2\mu)}$$

令
$$L=\frac{n^3(\lambda+2\mu)\left[M'+\dfrac{qz^2}{4h(\lambda+2\mu)}\right]+\dfrac{3q\lambda n}{2h(\lambda+2\mu)}}{-\left(\dfrac{3}{2h}-\dfrac{2z^2}{h^2}\right)}$$

$$m^2=\frac{6}{3h^2-4z^2}$$

得
$$\nabla^2 Q_x+m^2 Q_x=L\frac{\sin(nx+\delta)}{\cos(nx+\delta)+\cos(ny+\delta)}\tag{40}$$

$$\nabla^2 Q_x+m^2 Q_y=L\frac{\sin(ny+\delta)}{\cos(nx+\delta)+\cos(ny+\delta)}\tag{41}$$

对（40）的齐次二阶偏微分方程的通解为

$$Q_x=H_1\cos(mx+\delta_1)+H_1\cos(my+\delta_1)\tag{42}$$

上式 H_1 及 δ_1 为积分常数

令其特解为形式

$$Q_x = BL\frac{\sin(nx+\delta)}{\cos(nx+\delta)+\cos(ny+\delta)}$$

$$\frac{\partial Q_x}{\partial x} = BL\frac{n\cos(nx+\delta)\left\{\cos(nx+\delta)+\cos(ny+\delta)-\sin(nx+\delta)\left[-n\sin(nx+\delta)\right]\right\}}{\left[\cos(nx+\delta+\cos(ny+\delta)\right]^2}$$

$$= BLn\frac{1+\cos(nx+\delta)\cos(ny+\delta)}{\left[\cos(nx+\delta+\cos(ny+\delta)\right]^2}$$

$$\frac{\partial^2 Q_x}{\partial x^2} = BLn^2\sin(nx+\delta)\frac{3\cos(nx+\delta)\cos(ny+\delta)+\cos^2(ny+\delta)+2}{\left[\cos(nx+\delta)+\cos(ny+\delta)\right]^3}$$

$$\frac{\partial Q_x}{\partial y} = BL\sin(nx+\delta)\frac{\sin(n-ny+\delta)}{\left[\cos(nx+\delta)+\cos(ny+\delta)\right]^2}$$

$$\frac{\partial^2 Q_x}{\partial y^2} = -BLn^2\sin(nx+\delta)\frac{\cos(nx+\delta)\cos(ny+\delta)+\cos^2(ny+\delta)+2\sin^2(ny+\delta)}{\left[\cos(nx+\delta)+\cos(ny+\delta)\right]^3}$$

代入(40)，确定B的函数为

$$B = \frac{1}{\dfrac{2n^2\cos(ny+\delta)}{\cos(nx+\delta)+\cos(ny+\delta)}+m^2} = \frac{\cos(nx+\delta)+\cos(ny+\delta)}{(2n^2+m^2)\cos(ny+\delta)+m^2\cos(nx+\delta)}$$

$$\therefore\ Q_x = H_1\cos(mx+\delta_1)+H_1\cos(my+\delta_1)+L\frac{\sin(nx+\delta)}{(2n^2+m^2)\cos(ny+\delta)+m^2\cos(nx+\delta)}$$

由(42)，同理可得

$$Q_y = H_1\cos(mx+\delta_1)+H_1\cos(my+\delta_1)+L\frac{\sin(ny+\delta)}{(2n^2+m^2)\cos(nx+\delta)+m^2\cos(my+\delta)}$$

$\omega_0(x_1y)$及Q_x、Q_y全部解出，其中积分常数，可以由板的边界条件来确定之。

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