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用 C_n 群解梁弯曲问题

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摘要　应用 C_n 群的表示理论, 将分布多项式分解为正交特征分布多项式. 周期函数采用正交特征多项式逼近. 将均匀梁结构延拓并加上附加载荷, 使梁的位移化为周期函数. 利用正交特征分布多项式, 逼近梁的位移函数. 应用能量法求在载荷作用下梁位移的各个多项式的系数, 通过边界条件确定附加载荷的大小. 所述方法基本上为有限元方法, 其逼近精度与有限元是一致的. 此方法在求解中应用正交函数, 因而大大减低有限元的计算量. 该计算量与边界元相仿.

关键词　群, 有限元, 弯曲问题, 正交函数

中图分类号　TB 12; TB 111　　　文献标识码　A

对于梁弯曲问题研究有大量的文献, 有限元逼近是一种有效广泛应用的方法. 但是对于求解多节点有限元, 必须求解一个阶数很大的矩阵. 如何提高计算速度和降低计算量是有限元研究的一个重要方向. 钟万勰等曾将 C_{nv} 应用于对称结构问题的求解, 计算表明可以大大降低计算量[1, 2]. 目前用群论方法研究对称结构问题有大量的文献[3~10]. 我们也曾研究过应用群论构造正交有限元函数[9, 10]. 本文用群上空间算符的特征向量方法重新推导 C_n 群的正交基, 并将它应用于均匀梁弯曲问题的分析.

1　基本理论

定义局域函数 $f(x)$, $g(x)$ 为

$$\left.\begin{aligned} f(x) &= \begin{cases} 3(1-x)^2-2(1-x)^3, & 0\leqslant x\leqslant 1,\\ 3(1+x)^2-2(1+x)^3, & -1\leqslant x\leqslant 0,\end{cases}\\ f(x) &= \begin{cases} (1-x)^2-(1-x)^3, & 0\leqslant x\leqslant 1,\\ -(1+x)^2+(1+x)^3, & -1\leqslant x\leqslant 0.\end{cases}\end{aligned}\right\} \tag{1}$$

设周期区域为 $0\leqslant x\leqslant L$, 并利用平移算子 C_n^i, 其角标 $i=0, 1, 2, \cdots, n-1$. 将定义局域 $(0\leqslant x\leqslant L/n)$ 的函数 $f_n(x)=f(nx/L)$, $g_n(x)=g(nx/L)$, 扩展到整个周期区域的 n 个基函数. 这里 $C_n^i\varphi(x)=\varphi(x-iL/n)$, 并有

$$C_n^i f_n(x)=f_n(x-iL/n),\qquad C_n^i g_n(x)=g_n(x-iL/n), \tag{2}$$

其中 $C_n^i\varphi(x)=\varphi(x-iL/n)$. 构造有限元空间为

$$V_n=\{\sum_i(af_n(x-iL/n)+b_ig_n(x-iL/n)+b_ig_n(x-iL/n)\},\qquad \forall a_i,\ b_i\in IR. \tag{3}$$

考虑阿贝尔群 $C_n=(e, C_n^1, \cdots, C_n^{n-1})$ 和算子 $R=C_n^1+C_n^{n-1}$, 并应用特征向量求解方法, 得

$$RQ_\lambda=\lambda Q_\lambda. \tag{4}$$

可以得到群上空间的正交基为

$$Q_i=\sum_i d_i^k C_n^k,\qquad \overline{Q}_i=\sum_i \overline{d}_i^k C_n^k, \tag{5}$$

其中 $d_i^k=\cos(\frac{2ik}{n})$, $\overline{d}_i^k=\sin(\frac{2ik}{n})$, $i=0, 1, 2, \cdots, n/2$, $k=0, 1, 2, \cdots, n-1$.

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将阿贝尔群的正交基, 作用于有限元局部基函数在周期区域中$(0 \leqslant x \leqslant L)$. 我们在有限元空间中找到正交基为

$$\left.\begin{aligned}\Psi_i(x) &= Q_i f_n(x) = \sum_k d_i^k C_n^k f_n(x) = \sum_k d_i^k f_n\left(x - \frac{kL}{n}\right),\\ \varphi_n(x) &= Q_i g_n(x) = \sum_k d_i^k C_n^k g_n(x) = \sum_k d_i^k g_n\left(x - \frac{kL}{n}\right),\\ \overline{\Psi}_i(x) &= \overline{Q}_i f_n(x) = \sum_k \overline{d}_i^k C_n^k f_n(x) = \sum_k \overline{d}_i^k f_n\left(x - \frac{kL}{n}\right),\\ \overline{\varphi}_n(x) &= \overline{Q}_i g_n(x) = \sum_k \overline{d}_i^k C_n^k g_n(x) = \sum_k \overline{d}_i^k g_n\left(x - \frac{kL}{n}\right).\end{aligned}\right\} \tag{6}$$

其中 $i = 0, 1, 2, \cdots, n/2$.

为了方便, 我们设

$$\boldsymbol{F}_i = [\Psi_i, \varphi_i]^{\mathrm{T}}, \qquad \boldsymbol{G}_i = [\overline{\Psi}_i, \overline{\varphi}_i]^{\mathrm{T}}, \qquad \boldsymbol{f}_n = [f_n, g_n]^{\mathrm{T}}.$$

于是, 此式又可写成

$$\boldsymbol{F}_i = \sum d_i^k \boldsymbol{f}_n\left(x - \frac{kL}{n}\right), \qquad \boldsymbol{G}_i = \sum \overline{d}_i^k \boldsymbol{f}_n\left(x - \frac{kL}{n}\right). \tag{7}$$

上述函数满足

$$\left.\begin{aligned}&\int_0^L (\boldsymbol{U}_k)^{\mathrm{T}} \boldsymbol{V}_l \mathrm{d}x = 0, \quad \int_0^L (\boldsymbol{U}_k)^{\mathrm{T}} \boldsymbol{U}_l \mathrm{d}x = 0, \quad \int_0^L (\boldsymbol{V}_k)^{\mathrm{T}} \boldsymbol{V}_l \mathrm{d}x = 0, \quad k \neq l,\\ &\int_0^L (\boldsymbol{U}_k)^{\mathrm{T}} \boldsymbol{V}_l \mathrm{d}x = 0, \quad \int_0^L (\boldsymbol{U}_k)^{\mathrm{T}} \boldsymbol{U}_l \mathrm{d}x \neq 0, \quad \int_0^L (\boldsymbol{V}_k)^{\mathrm{T}} \boldsymbol{V}_l \mathrm{d}x \neq 0, \quad k = l.\end{aligned}\right\} \tag{8}$$

2　在梁弯曲分析中的应用

一般梁结构皆为非周期. 将梁结构延拓并加上附加载荷, 使成为周期结构. 于是, 梁位移写成

$$w(x) = \sum_i (\boldsymbol{a}_i Q_i \boldsymbol{f}_n + \overline{\boldsymbol{a}}_i \overline{Q}_i \boldsymbol{f}_n). \tag{9}$$

在式(9)中, $\boldsymbol{a}_i = [a_{i,1}, a_{i,2}]$, $\overline{\boldsymbol{a}}_i = [\overline{a}_{i,1}, \overline{a}_{i,2}]$, 其中 $\boldsymbol{a}_i$ 为待定常数.

在每一个特征函数子空间, 内转角为

$$\theta_i = Q_i \boldsymbol{a}_i \frac{\partial(\boldsymbol{f}_n)}{\partial x}, \qquad \overline{\theta}_i = \overline{Q}_i \overline{\boldsymbol{a}}_i \frac{\partial(\boldsymbol{f}_n)}{\partial x}, \qquad i = 0, 1, 2, \cdots, n/2. \tag{10}$$

在每一个特征函数子空间内, 屈率为

$$\zeta_i = Q_i \boldsymbol{a}_i \frac{\partial^2(\boldsymbol{f}_n)}{\partial x^2}, \qquad \overline{\zeta}_i = \overline{Q}_i \overline{\boldsymbol{a}}_i \frac{\partial^2(\boldsymbol{f}_n)}{\partial x^2}, \qquad i = 0, 1, 2, \cdots, n/2. \tag{11}$$

每个特征小空间的应变能为

$$\begin{aligned}\boldsymbol{U}_i = {}& \frac{EI}{2}\int_0^L (\zeta_i)^2 \mathrm{d}x = \\ & \frac{EI}{2}\left(\int_0^L \boldsymbol{a}_i \left(Q_i \frac{\partial^2 \boldsymbol{f}_n}{\partial x^2}\right)\left(Q_i \frac{\partial^2 \boldsymbol{f}_n^{\mathrm{T}}}{\partial x^2}\right) \boldsymbol{a}_i^{\mathrm{T}} \mathrm{d}x + \int_0^L \boldsymbol{a}_i \left(Q_i \frac{\partial^2 \boldsymbol{f}_n}{\partial x^2}\right)\left(\overline{Q}_i \frac{\partial^2 \boldsymbol{f}_n^{\mathrm{T}}}{\partial x^2}\right) \overline{\boldsymbol{a}}_i^{\mathrm{T}} \mathrm{d}x + \right.\\ & \left.\int_0^L \overline{\boldsymbol{a}}_i \left(\overline{Q}_i \frac{\partial^2 \boldsymbol{f}_n}{\partial x^2}\right)\left(Q_i \frac{\partial^2 \boldsymbol{f}_n^{\mathrm{T}}}{\partial x^2}\right) \boldsymbol{a}_i^{\mathrm{T}} \mathrm{d}x + \int_0^L \overline{\boldsymbol{a}}_i \left(\overline{Q}_i \frac{\partial^2 \boldsymbol{f}_n}{\partial x^2}\right), \left(\overline{Q}_i \frac{\partial^2 \boldsymbol{f}_n^{\mathrm{T}}}{\partial x^2}\right) \overline{\boldsymbol{a}}_i^{\mathrm{T}} \mathrm{d}x\right) \\ & \left(\frac{n}{L}\right)^4 \frac{EIL}{4}\left(\boldsymbol{a}_i\left(\sum_{s=0}^{2} \cos\left(\frac{2is\pi}{n}\right) \int_0^2 \frac{\partial^2 \boldsymbol{f}(x-1)}{\partial x^2} \frac{\partial^2 \boldsymbol{f}^{\mathrm{T}}(x-s)}{\partial x^2} \mathrm{d}x\right) \boldsymbol{a}_i^{\mathrm{T}} + \right.\\ & - \boldsymbol{a}_i\left(\sum_{s=0}^{2} \sin\left(\frac{2is\pi}{n}\right) \int_0^2 \frac{\partial^2 \boldsymbol{f}(x-1)}{\partial x^2} \frac{\partial^2 \boldsymbol{f}^{\mathrm{T}}(x-s)}{\partial x^2} \mathrm{d}x\right) \overline{\boldsymbol{a}}_i^{\mathrm{T}} + \\ & - \overline{\boldsymbol{a}}_i\left(\sum_{s=0}^{2} \sin\left(\frac{2is\pi}{n}\right) \int_0^2 \frac{\partial^2 \boldsymbol{f}(x-1)}{\partial x^2} \frac{\partial^2 \boldsymbol{f}^{\mathrm{T}}(x-s)}{\partial x^2} \mathrm{d}x\right) \boldsymbol{a}_i^{\mathrm{T}} + \\ & \left.\overline{\boldsymbol{a}}_i\left(\sum_{s=0}^{2} \cos\left(\frac{2is\pi}{n}\right) \int_0^2 \frac{\partial^2 \boldsymbol{f}(x-1)}{\partial x^2} \frac{\partial^2 \boldsymbol{f}^{\mathrm{T}}(x-s)}{\partial x^2} \mathrm{d}x\right) \overline{\boldsymbol{a}}_i^{\mathrm{T}}\right) = \\ & \left(\frac{n}{L}\right)^4 \frac{EIL}{4}\left(\boldsymbol{a}_i \boldsymbol{Q}_i^{11} \boldsymbol{a}_i^{\mathrm{T}} + \boldsymbol{a}_i \boldsymbol{Q}_i^{12} \overline{\boldsymbol{a}}_i^{\mathrm{T}} + \overline{\boldsymbol{a}}_i \boldsymbol{Q}_i^{21} \boldsymbol{a}_i^{\mathrm{T}} + \overline{\boldsymbol{a}}_i \boldsymbol{Q}_i^{22} \overline{\boldsymbol{a}}_i^{\mathrm{T}}\right).\end{aligned} \tag{12}$$

这里

$$\left.\begin{aligned}\boldsymbol{Q}_i^{11} = \boldsymbol{Q}_i^{22} = \sum_{s=0}^{2}\cos\left(\frac{2is\pi}{n}\right)\int_0^2 \frac{\partial^2 \boldsymbol{f}(x-1)}{\partial x^2}\frac{\partial^2 \boldsymbol{f}^{\mathrm{T}}(x-s)}{\partial x^2}\mathrm{d}x,\\ \boldsymbol{Q}_i^{21} = \boldsymbol{Q}_i^{12} = -\sum_{s=0}^{2}\sin\left(\frac{2is\pi}{n}\right)\int_0^2 \frac{\partial^2 \boldsymbol{f}(x-1)}{\partial x^2}\frac{\partial^2 \boldsymbol{f}^{\mathrm{T}}(x-s)}{\partial x^2}\mathrm{d}x.\end{aligned}\right\} \tag{13}$$

外力功为

$$\begin{aligned} W = & \int(\boldsymbol{a}_i Q\boldsymbol{f}_n + \bar{\boldsymbol{a}}_i\bar{Q}\boldsymbol{f}_n)\,q\,\mathrm{d}x + \sum_k\int(\boldsymbol{a}_i Q\boldsymbol{f}_n + \bar{\boldsymbol{a}}_i\bar{Q}\boldsymbol{f}_n)P_k(x)\,\delta(x_k)\,\mathrm{d}x + \\ & \sum_k\int(\boldsymbol{a}_i Q\boldsymbol{f}_n + \bar{\boldsymbol{a}}_i\bar{Q}\boldsymbol{f}_n)M_k(x)\,\delta(x_k)\,\mathrm{d}x = \\ & \boldsymbol{a}_i\int Q\boldsymbol{f}_n(q + \Sigma P(x_k)\,\delta(x_k)) + Q_i\frac{\partial \boldsymbol{f}_n}{\partial x}\Sigma M(x_k)\,\delta(x_k)\,\mathrm{d}x + \\ & \bar{\boldsymbol{a}}_i\int\bar{Q}\boldsymbol{f}(q + \Sigma P(x_k)\,\delta(x_k)) + \bar{Q}_i\frac{\partial \boldsymbol{f}_n}{\partial x}\Sigma M(x_k)\,\delta(x_k)\,\mathrm{d}x = \boldsymbol{a}_i\boldsymbol{W}_i + \bar{\boldsymbol{a}}_i\bar{\boldsymbol{W}}_i, \end{aligned} \tag{14}$$

其中 $i = 0, 1, 2, \cdots, n/2$. 这里

$$\left.\begin{aligned}\boldsymbol{W}_i = \int Q\boldsymbol{f}_n(q + \Sigma P(x_k)\,\delta(x_k)) + Q_i\frac{\partial \boldsymbol{f}_n}{\partial x}\Sigma M(x_k)\,\delta(x_k)\,\mathrm{d}x,\\ \bar{\boldsymbol{W}}_i = \int \bar{Q}\boldsymbol{f}_n(q + \Sigma P(x_k)\,\delta(x_k)) + \bar{Q}_i\frac{\partial \boldsymbol{f}_n}{\partial x}\Sigma M(x_k)\,\delta(x_k)\,\mathrm{d}x.\end{aligned}\right\} \tag{13}$$

如果集中力与集中力矩作用在接点上, 根据 $F(X)$, $G(X)$ 的定义, 上式可以写成

$$\left.\begin{aligned}\boldsymbol{W}_i = \int Q\boldsymbol{f}_n q\,\mathrm{d}x + \sum_k\left(\boldsymbol{P}_k d_i^k\begin{bmatrix}1\\0\end{bmatrix} + \frac{n}{L}\boldsymbol{M}_k d_i^k\begin{bmatrix}0\\1\end{bmatrix}\right),\\ \boldsymbol{W}_i = \int \bar{Q}\boldsymbol{f}_n q\,\mathrm{d}x + \sum_k\left(\boldsymbol{P}_k \bar{d}_i^k\begin{bmatrix}1\\0\end{bmatrix} + \frac{n}{L}\boldsymbol{M}_k \bar{d}_i^k\begin{bmatrix}0\\1\end{bmatrix}\right).\end{aligned}\right\} \tag{16}$$

在子空间上应用能量法原理有

$$\frac{\partial \boldsymbol{U}_i}{\partial \boldsymbol{a}_i} = \boldsymbol{W}_i, \qquad \frac{\partial \boldsymbol{U}_i}{\partial \bar{\boldsymbol{a}}_i} = \bar{\boldsymbol{W}}_i, \qquad i = 1, 2, \cdots, n/2. \tag{17}$$

上式可以写成

$$\left.\begin{aligned}(\boldsymbol{Q}_i^{11}\boldsymbol{a}_i^{\mathrm{T}} + \boldsymbol{Q}_i^{12}\bar{\boldsymbol{a}}_i^{\mathrm{T}})\left(\frac{n}{L}\right)^4\frac{EIL}{2} = \boldsymbol{W}_i,\\ (\boldsymbol{Q}_i^{21}\boldsymbol{a}_i^{\mathrm{T}} + \boldsymbol{Q}_i^{22}\bar{\boldsymbol{a}}_i^{\mathrm{T}})\left(\frac{n}{L}\right)^4\frac{EIL}{2} = \bar{\boldsymbol{W}}_i.\end{aligned}\right\} \tag{18}$$

上述 E 为弹性模量, I 为抗弯截面惯性矩, L 为周期. 上式方程最多不超过 4 个未知数.

为了满足边界条件, 可以在其边界上加上附加的外力、力偶, 并设梁上作用力为 F. 附加力和力偶分别为 F_n 和 M_n, 有

$$w(x) = \sum_i(\boldsymbol{a}_i Q\boldsymbol{f}_n + \bar{\boldsymbol{a}}_i\bar{Q}\boldsymbol{f}_n). \tag{19}$$

由式(19)可以看出, 位移是附加力和力偶的线性函数. 梁的边界条件, 其中简支边界为 $w|_{x=x_i} = 0$, $M|_{x=x_i} = 0$; 自由边界为 $Q|_{x=x_i} = 0$, $M|_{x=x_i} = 0$; 固支边界为 $\theta|_{x=x_i} = 0$, $w|_{x=x_i} = 0$. 图 1 为简支梁受均布载荷作用. 将梁延拓(图 2), 并加上附加载荷(F_1, F_2, M_1, M_2). 由边界条件求出附加载荷, 计算结果如表 1 所示. 表中节点(n)位移 $w = 1l^4(12EI)^{-1}w_0$.

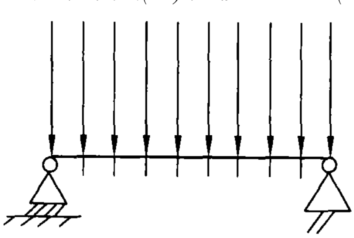

图 1 受均匀载荷作用的简支梁

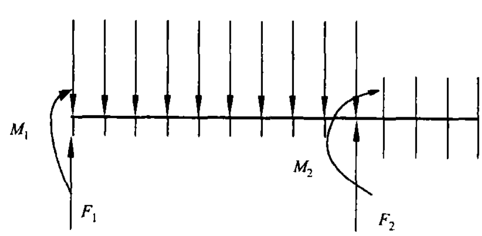

图 2 梁的延拓

表 1　梁位移量(w_0)计算结果对比

w_0	节点						
	2	3	4	5	6	7	8
计算值	0.057 128 90	0.105 468 70	0.137 207 03	0.148 437 50	0.137 207 03	0.105 468 70	0.057 128 90
理论值	0.060 668 00	0.111 328 10	0.144 653 00	0.156 250 00	0.144 653 00	0.111 328 10	0.060 668 00

3　结束语

应用 C_n 群将一维周期函数分解为正交函数基. 应用于梁弯曲问题求解, 由于每个正交基不超过 4 个未知量, 所以最大矩阵的大小由边界条件决定. 这种方法, 可以大大减少有限元的计算量. 应用群的乘积, 可实现本文方法向二维和三维问题的推广.

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Solving the Bending of Beam by Using C_n Group

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Abstract　The representation theory of C_n group is applied to the decomposition of distributed polynominals into distributed polynomial with characteristic of orthogonality, which is adopted for approximating periodic function. A homogeneous structure of beam is in continuation and bears additional load; and the displacement of the beam is converted into periodic function. Thus the authors use distributed polynomial with orthogonal characteristic for approximating displacement function of the beam; and apply energy method for solving coefficient of each polynomial reflecting dispacement of beam under the action of load; and draw support from boundary condition for determining the size of additional load. The method mentioned above is basically the finite element methoel with the same approximation accuracy. The method apply orthogonal functions in its solving. As a result, its amount of calculation is less than that of finite element method but is similar to that of buoundary element method.

Keywords　group, finite element, problem of bending, orthogonal function